CS103
Winter 2026

Lecture 00:

Intro to Set Theory

Are there “laws of physics” in computer science?

For this, along with further course motivation and Day 1 logistics, see the Prezi included on today’s lecture page.

Key Questions in CS103

- ***Computability Theory***
  - What problems can you solve with a computer?
- ***Complexity Theory***
  - Why are some problems harder to solve than others?
- ***Discrete Mathematics***
  - How can we be certain in our answers to these questions?

Course Website

https://cs103.stanford.edu

All course content will be hosted here, except for lecture videos.

CS103 ACE

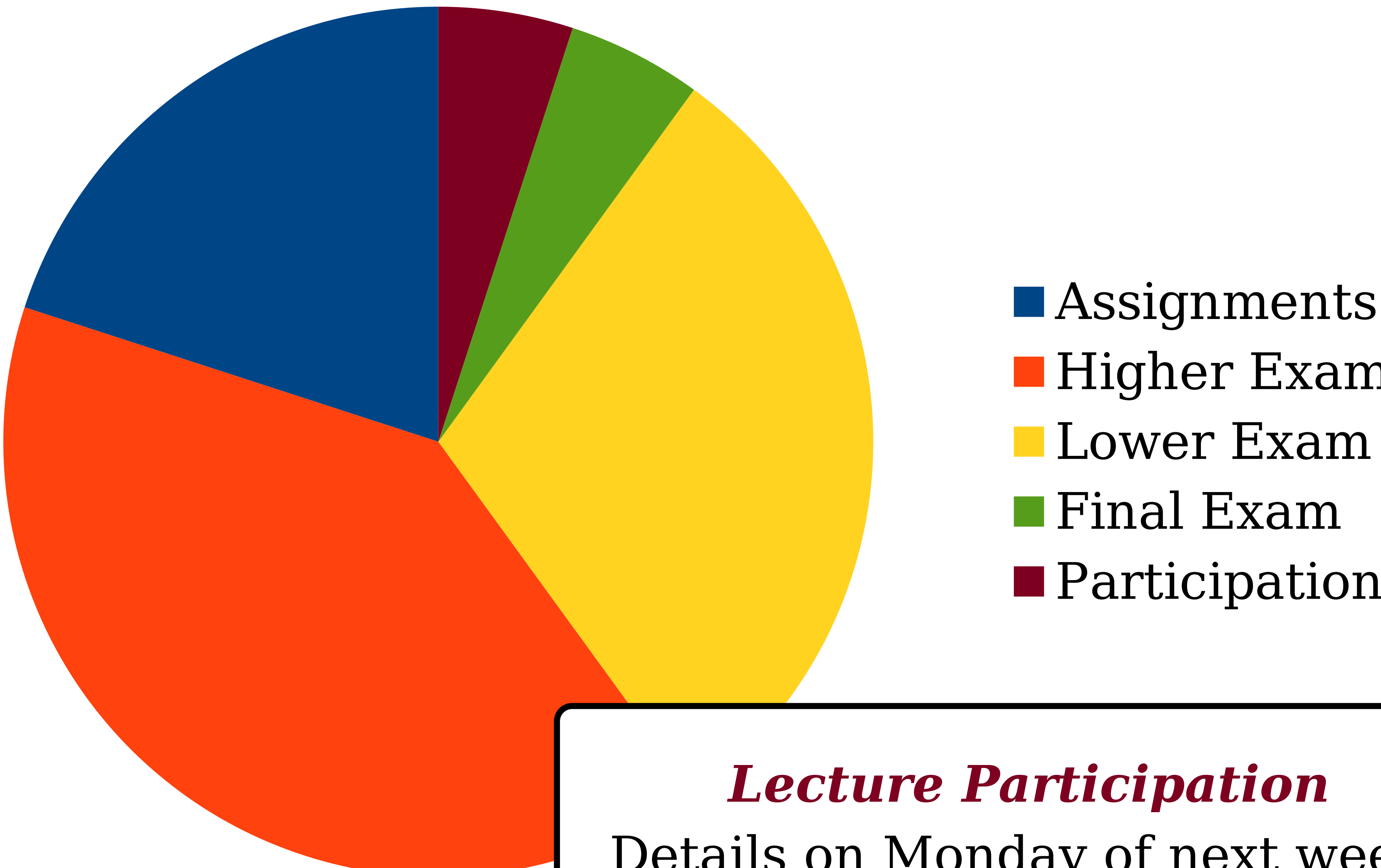
- ***CS103 ACE*** is an optional, one-unit companion course to CS103.
- CS103 ACE meets Tuesdays, 1:30 – 3:20 PM and provides additional practice with the course material in a small group setting.
- Interested? Apply online using ***this link.***

Evelyn Yee
(they/them)
ACE Instructor

Problem Set 0

- Your first assignment, Problem Set 0, goes out today. It’s due Friday at 1:00 PM Pacific.
- This assignment requires you to set up your development environment and to get set up on Gradescope and EdStem.
- There’s no coding involved, but it’s good to start early in case you encounter any technical issues.

Grading



Lecture Participation

Details on Monday of next week!

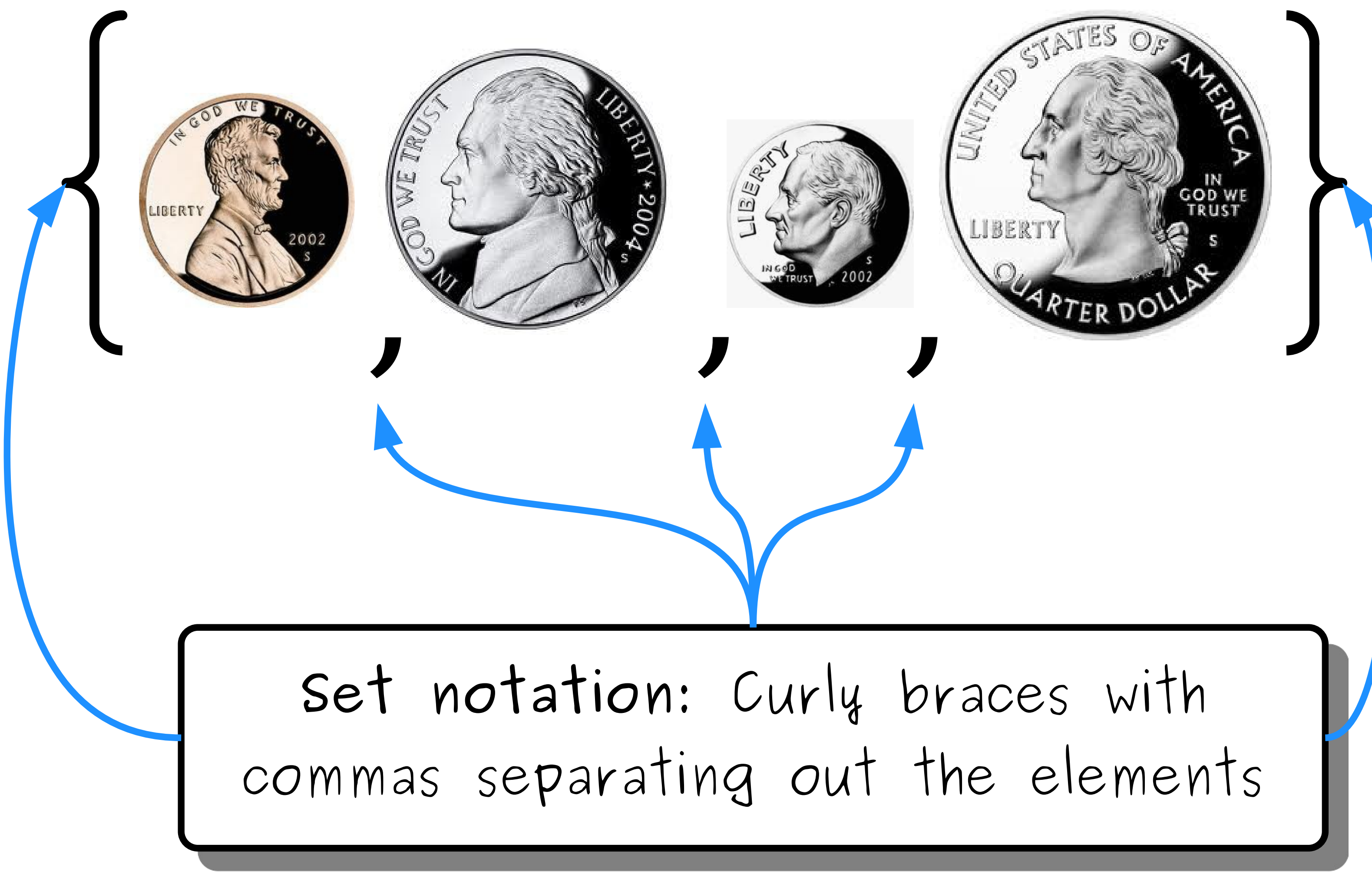
Approaching this Course

- Key recommendations:
  - ***Attend lecture in person***. This ensures you stay current with the concepts and gives time to digest the content.
  - ***Take notes by hand***. Explaining concepts in your own words improves learning and identifies questions to ask.
  - ***Ask questions!*** That's what we're here for. Feel free to ask in lecture, after class, on EdStem, or in office hours.
  - ***Read the Guides***. We post readings on the website in the form of "Guide to *X*." Those are the main readings for the course.
  - ***Focus on learning***. Problem sets are worth much less than the exams. Prioritize building skills over completion. Avoid shortcuts.
- CS103 is more like building a rocket than learning a language: there's less frequent feedback you'll need to review in more depth.

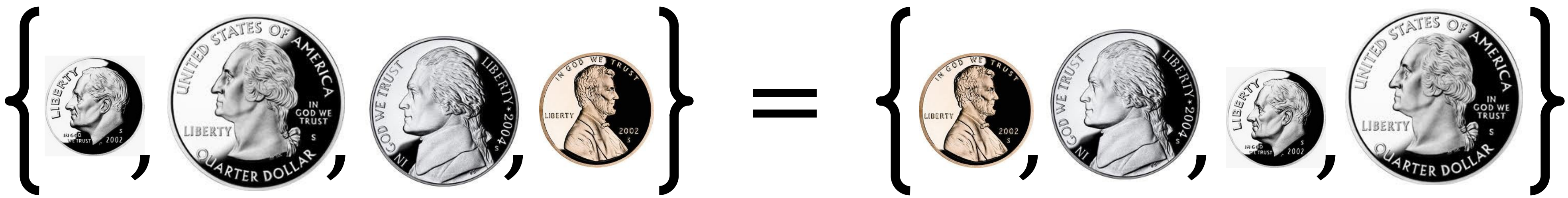
Let's get started on this incredible journey!

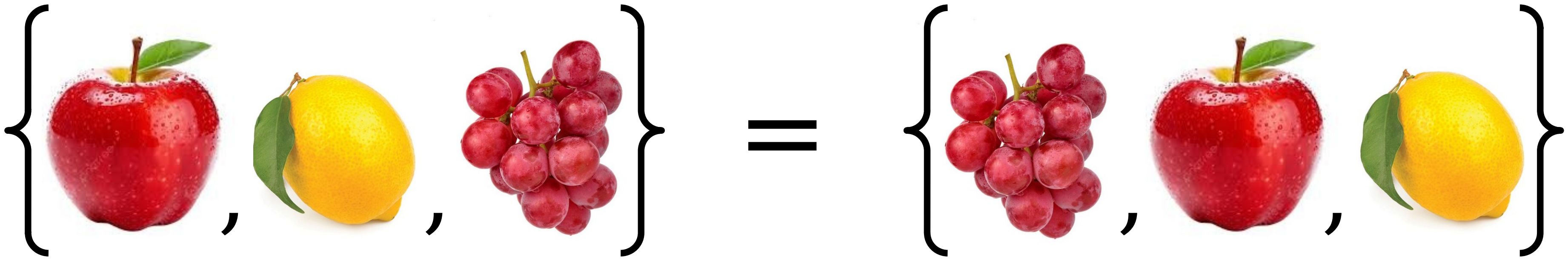
Introduction to Set Theory

1. **Sets and Elements (Definitions and Notation)**
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

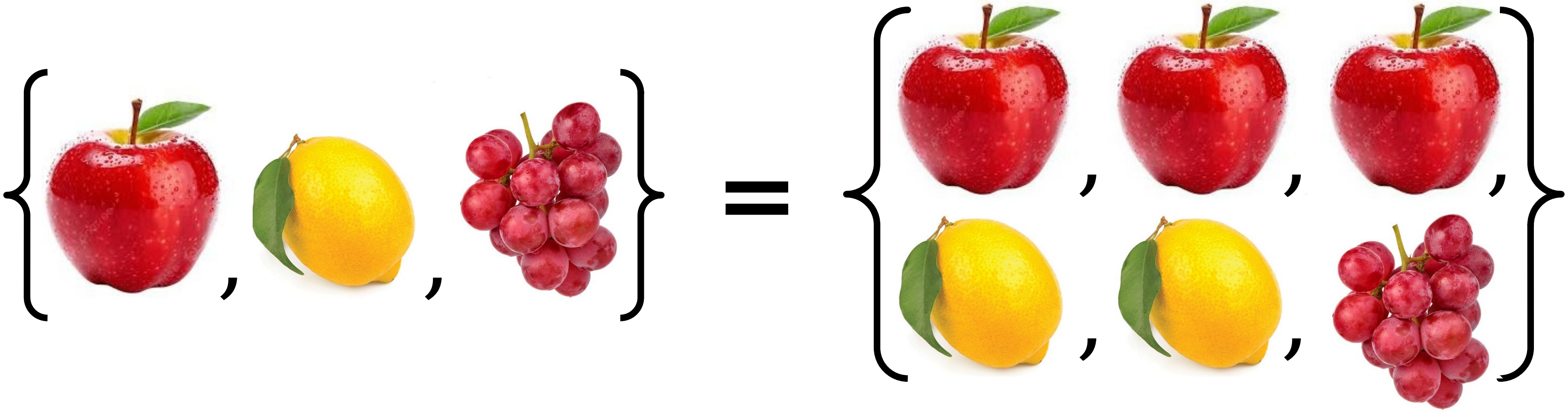


A ***set*** is an unordered collection of distinct objects, which may be anything, including other sets.



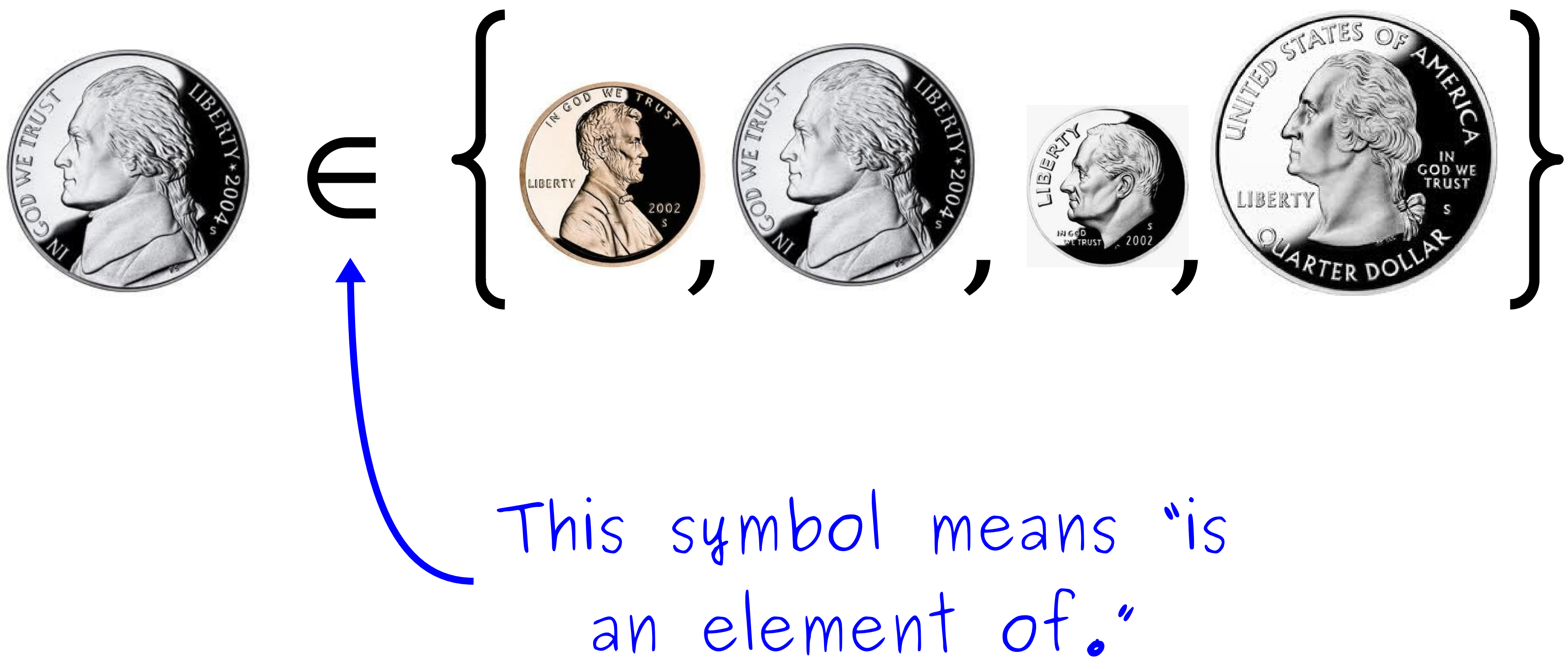


Two sets are equal when they have the
same contents, ignoring order.



These are two descriptions of the same set. On the right, the exact **same apple** is listed multiple times; same for the lemon.

Sets cannot contain duplicate elements.
Any repeated elements are ignored.

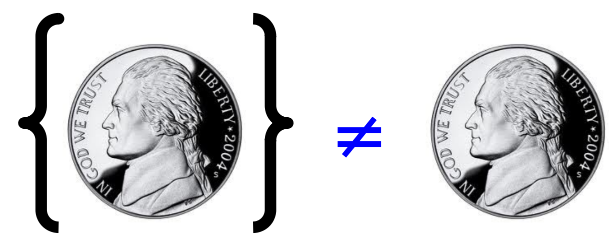


The objects that make up a set are called the ***elements*** of that set.

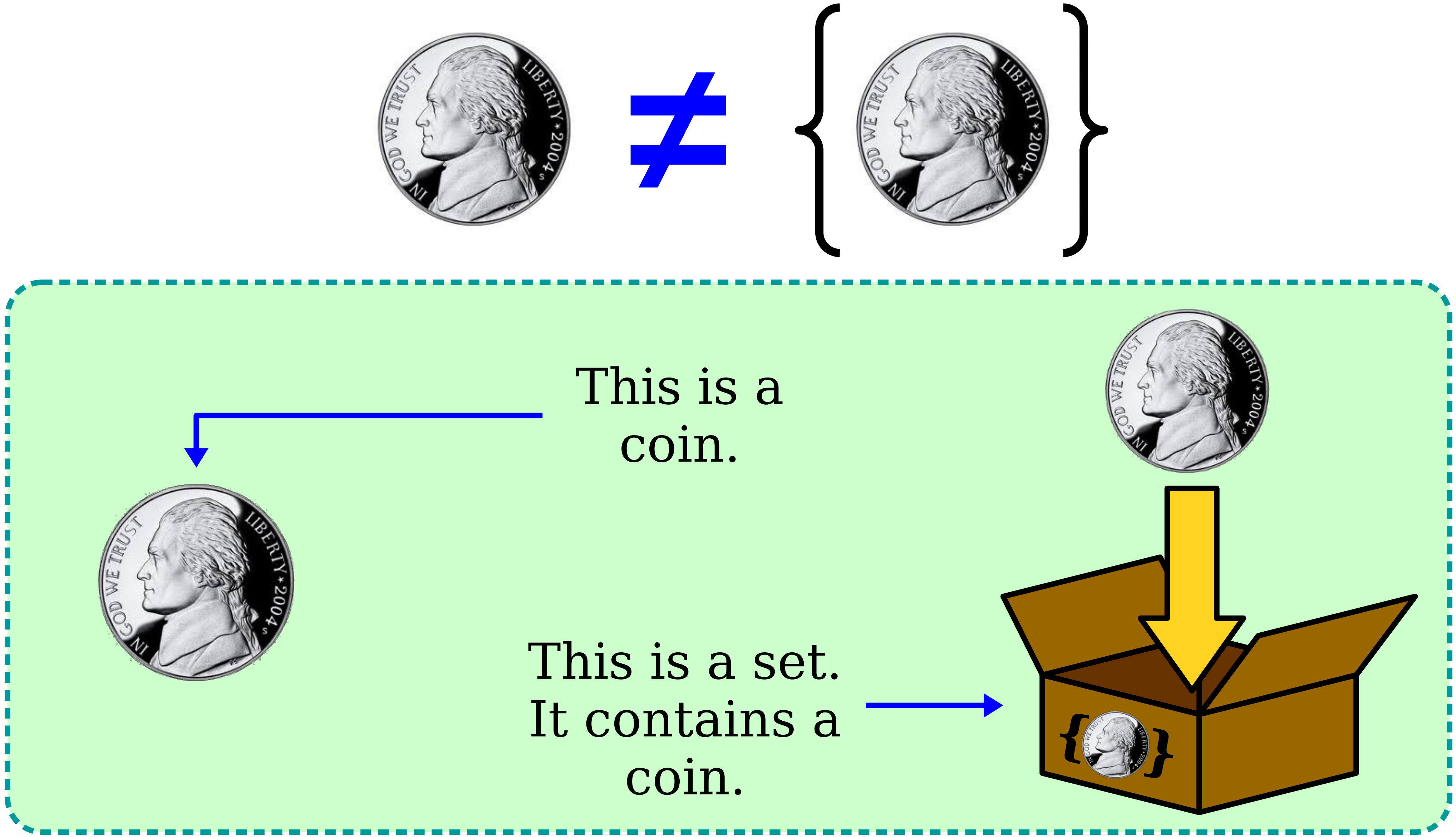
This symbol means "is not an element of."

The objects that make up a set are called the ***elements*** of that set.

Important distinction:

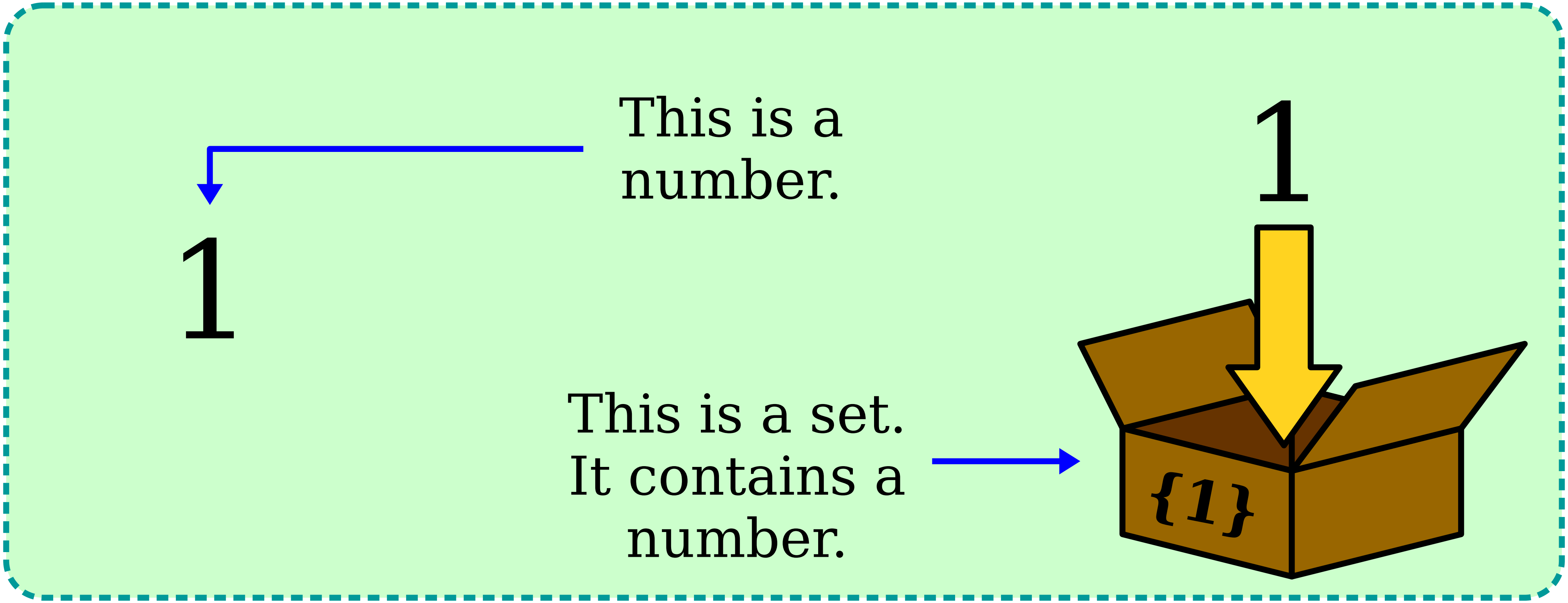


Question: Is this an element of the set?



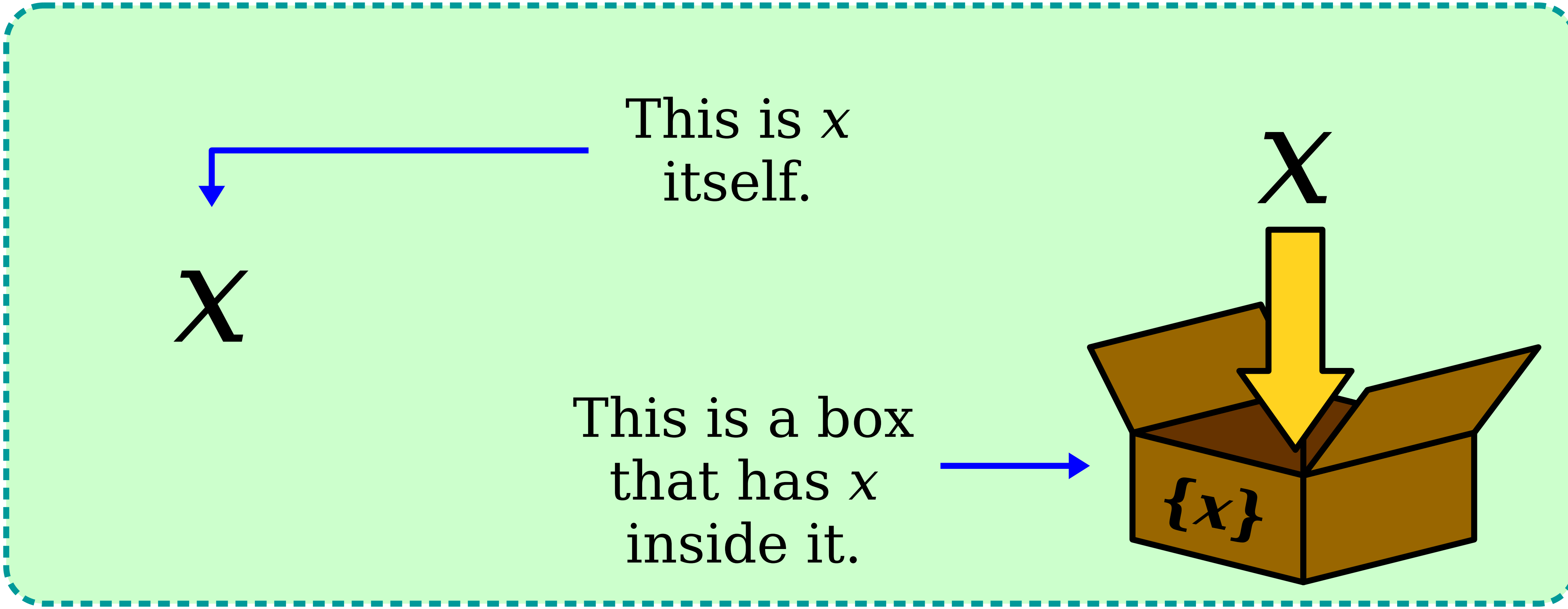
No object x is equal to the set containing x.

$$1 \neq \{1\}$$



No object x is equal to the set containing x.

$$x \neq \{x\}$$



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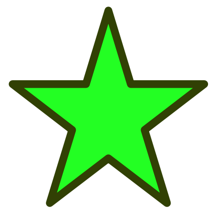
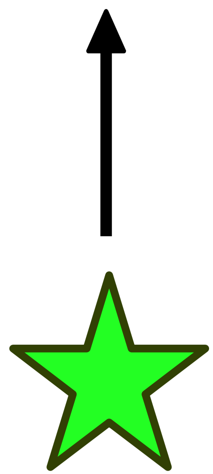
Subsets

- A set S is called a ***subset*** of a set T (denoted $\boldsymbol{S \subseteq T}$) when all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\{ c, b \} \subseteq \{ a, b, c, d \}$
  - {  ,  ,  } ⊆ {  ,  ,  }

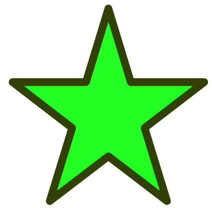
Elements and Subsets

$S = \{\, 2,$  $, \{2\},$  $,$ 

$\}$

 $\in S$

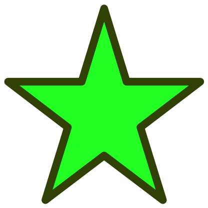
Elements and Subsets

$S = \{ 2,$  $, \{2\},$  $,$ 

$\}$

$$2 \in S$$

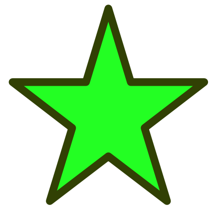
Elements and Subsets

$S = \{ 2,$  $, \{2\},$  $,$ 

$\}$

$$\{2\} \in S$$

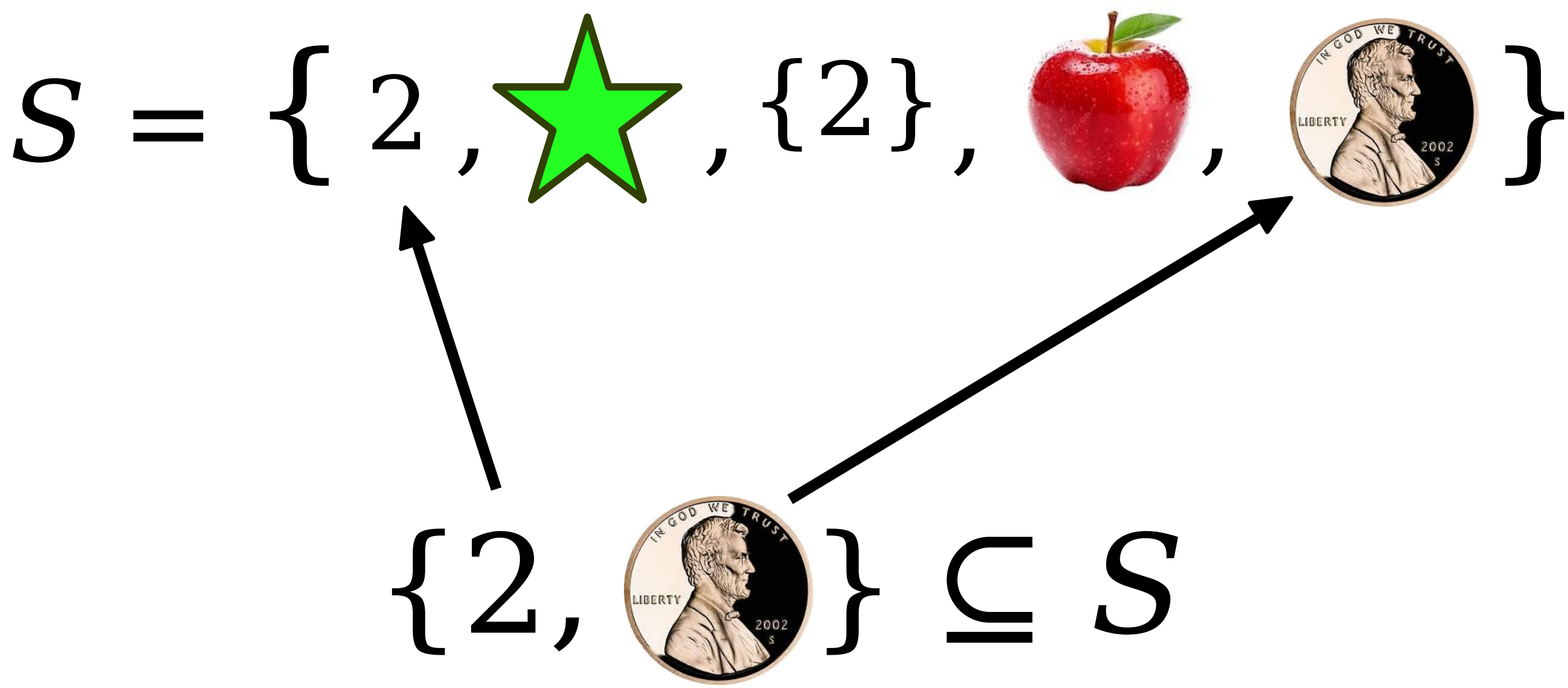
Elements and Subsets

$S = \{$ 2, , $\{2\}$, ,  $\}$

$\{2,$  $\} \notin S$

General intuition: $x \in S$ means you can ***point at x inside of S***.

Elements and Subsets



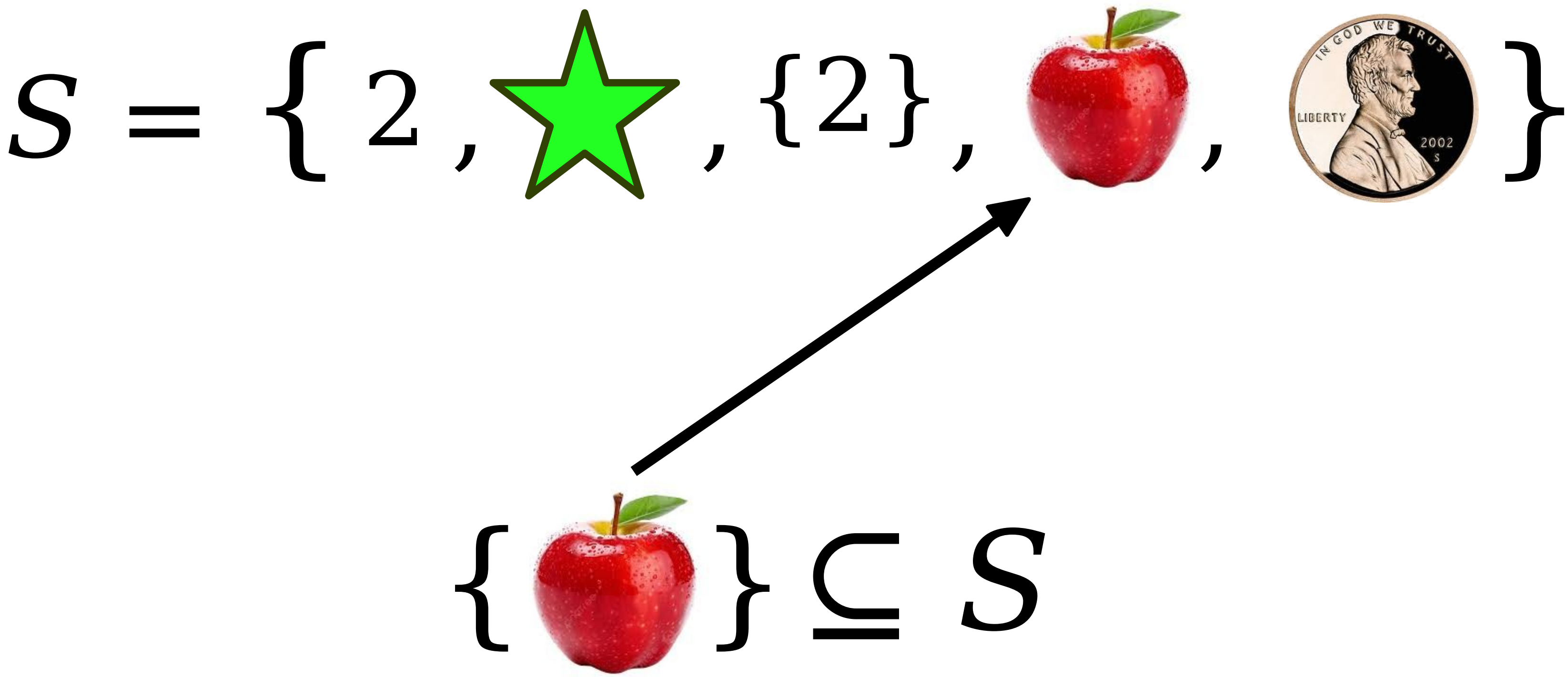
Elements and Subsets

$S = \{\ 2,$  $,\ \{2\},$  $,$ 

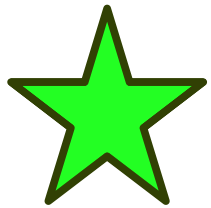
$\}$

$$\{2\} \subseteq S$$

Elements and Subsets



Elements and Subsets

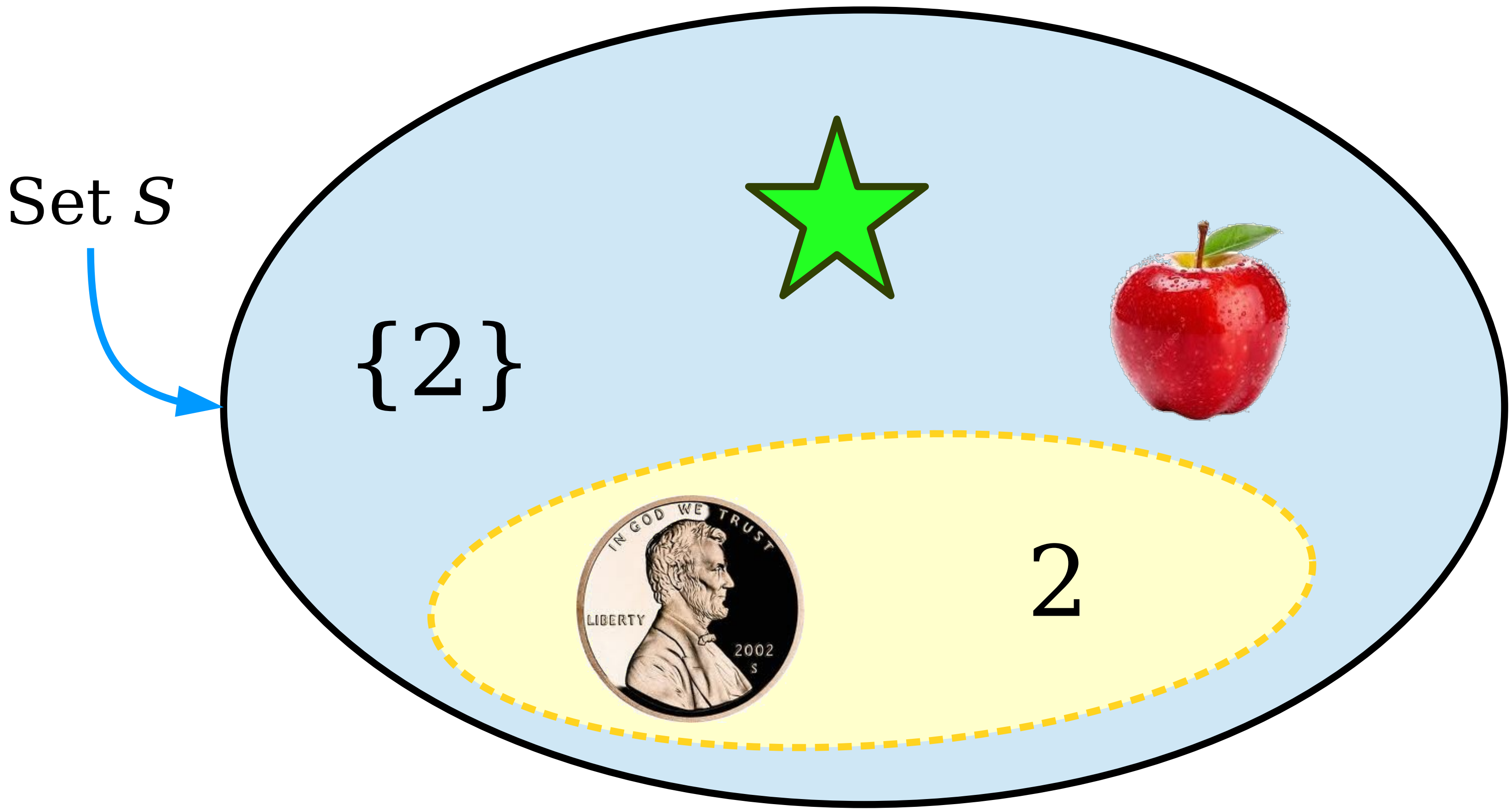
$S = \{ 2,$  $, \{2\},$  $,$ 

$\}$

$$2 \not\subseteq S$$

(Since 2 isn't a set.)

Elements and Subsets



$\{2,$  $\} \subseteq S$

General intuition:
$A \subseteq B$ if you can form A by ***circling elements of B***.

Elements and Subsets

- We say that $\boldsymbol{S \in T}$ when, among the elements of T, one of them is *exactly* the object S.
- We say that $\boldsymbol{S \subseteq T}$ when S is a set and every element of S is also an element of T. (S has to be a set for the statement $S \subseteq T$ to be true.)
- Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.
- We have a resource on the course website, the Guide to Elements and Subsets, that explores this in more depth.

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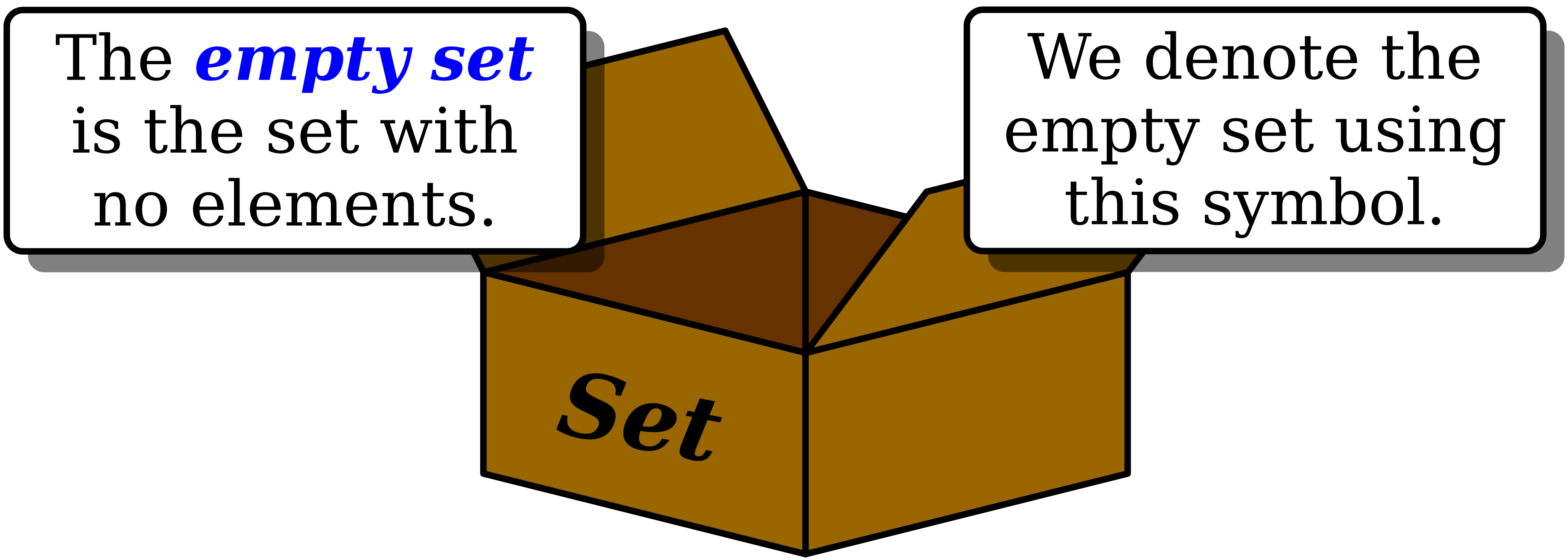
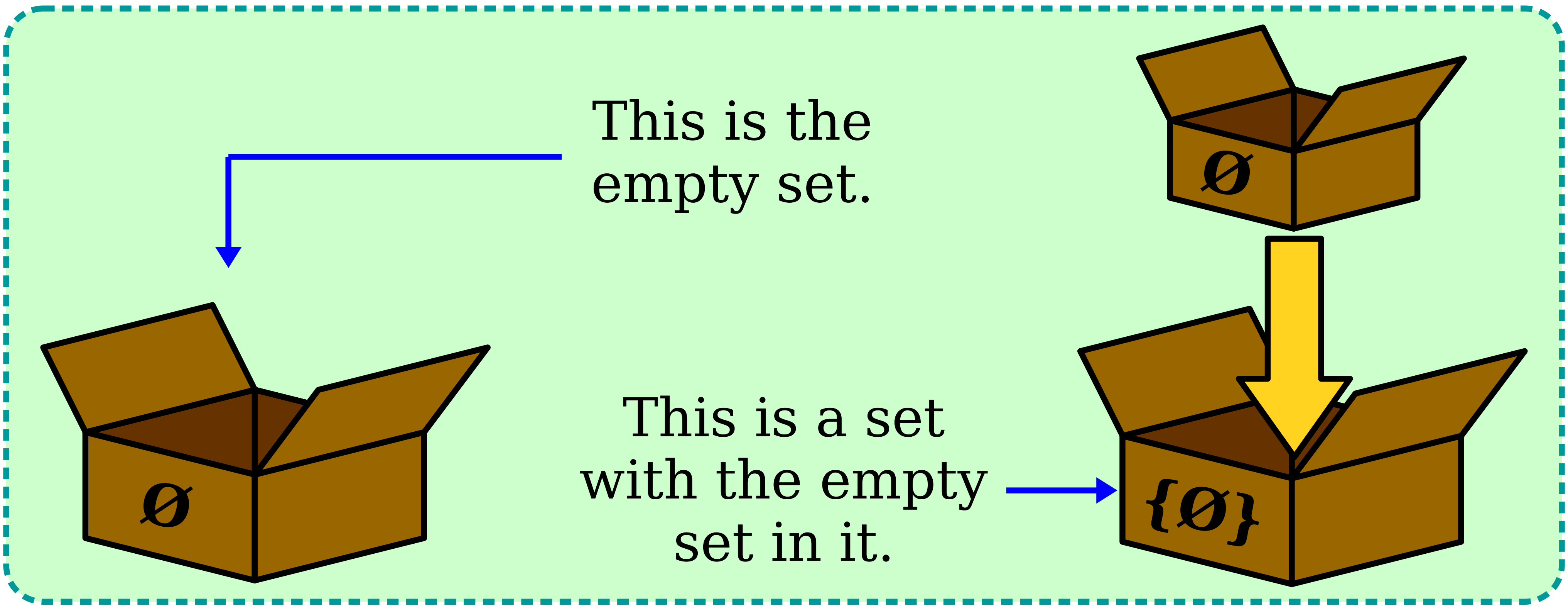
$$\{\ \} = \emptyset$$



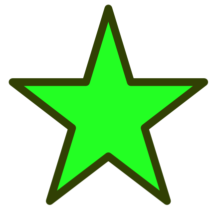
Sets can contain any number of elements.

$$\emptyset \neq \{\emptyset\}$$



No object x is equal to the set containing x.

Elements and Subsets

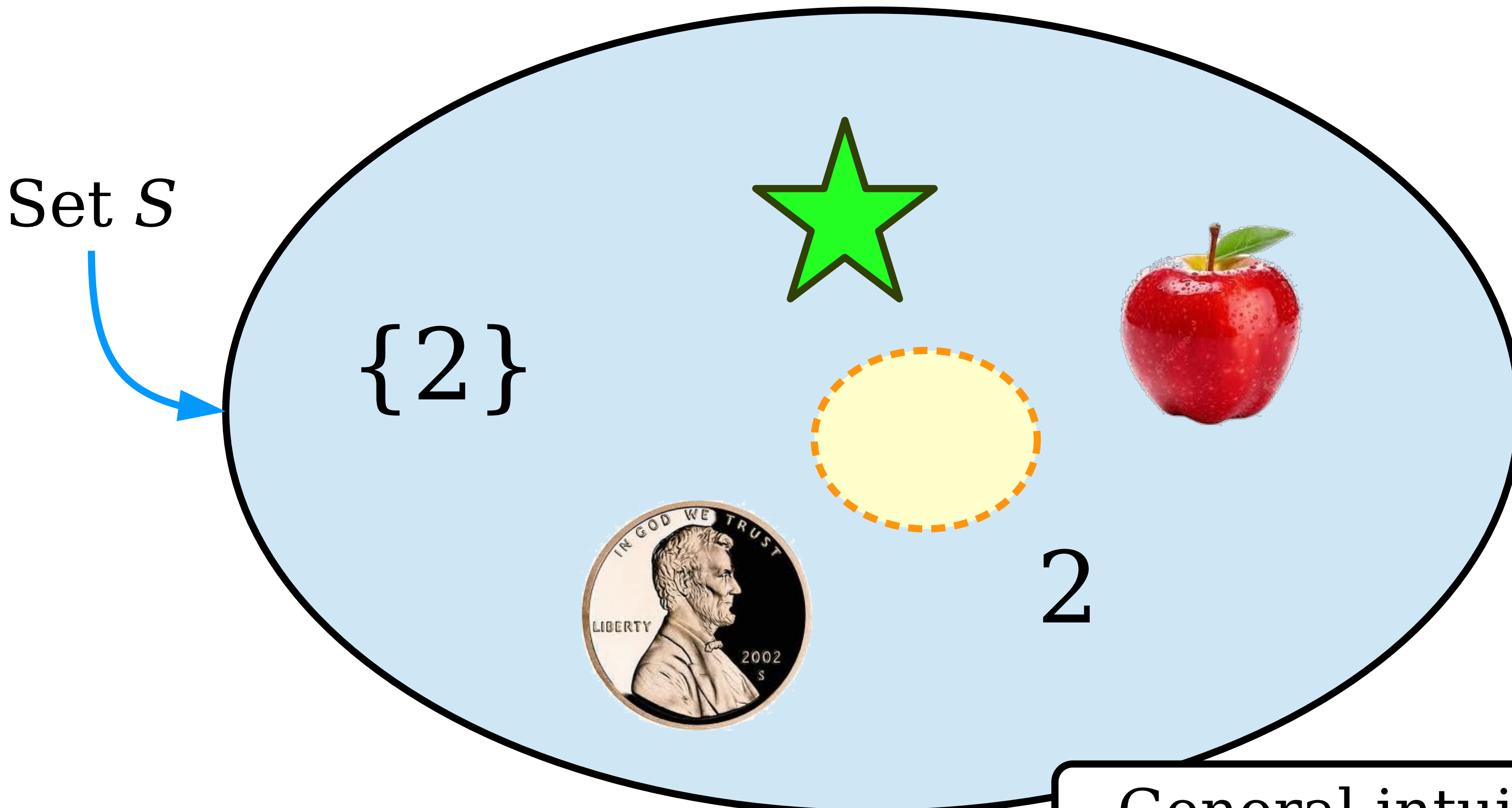
$S = \{$ 2, , $\{2\}$, , 

$\}$

$$\emptyset \notin S$$

General intuition: $x \in S$ means you can ***point at x inside of S***.

Elements and Subsets



$$\emptyset \subseteq S$$

General intuition: $A \subseteq B$ if you can form A by ***circling elements of B***.

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Infinite Sets

- Some sets contain *infinitely many* elements!
- The set ℕ = { 0, 1, 2, 3, …} is the set of all the ***natural numbers***.
  - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set ℤ = { …, -2, -1, 0, 1, 2, … } is the set of all the ***integers***.
  - Z is from German "Zahlen."
- The set ℝ is the set of all ***real numbers***.
  - $e \in \mathbb{R}$, and $4 \in \mathbb{R}$, and $-137 \in \mathbb{R}$.
- A few subset relationships:
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

Describing Complex Sets

- Here are some English descriptions of infinite sets:

  "The set of all even natural numbers."

  "The set of all real numbers less than 137."

  "The set of all Python programs."

- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

where

n is a natural number

and n is even

$$\{\ 0, 2, 4, 6, 8, 10, 12, 14, 16, \ldots\ \}$$

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{\ x \mid \textit{some property } x \textit{ satisfies}\ \}$$

$$\{\ x \in S \mid \textit{some property } x \textit{ satisfies}\ \}$$

- For example:

$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$

$\{\ C \mid C \text{ is a set of US coins}\ \}$

$\{\ r \in \mathbb{R} \mid r < 3\ \}$

$\{\ n \in \mathbb{N} \mid n < 3\ \}$ **(the set {0, 1, 2})**

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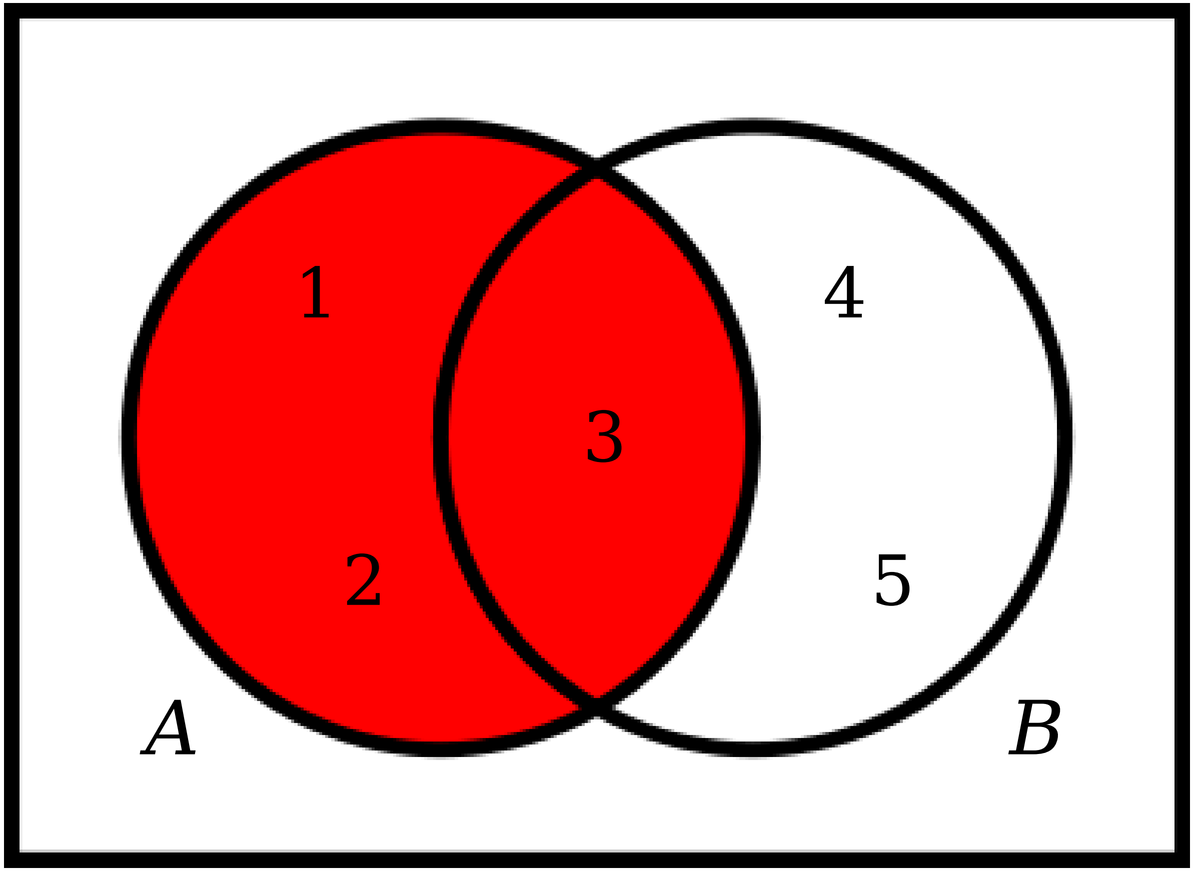
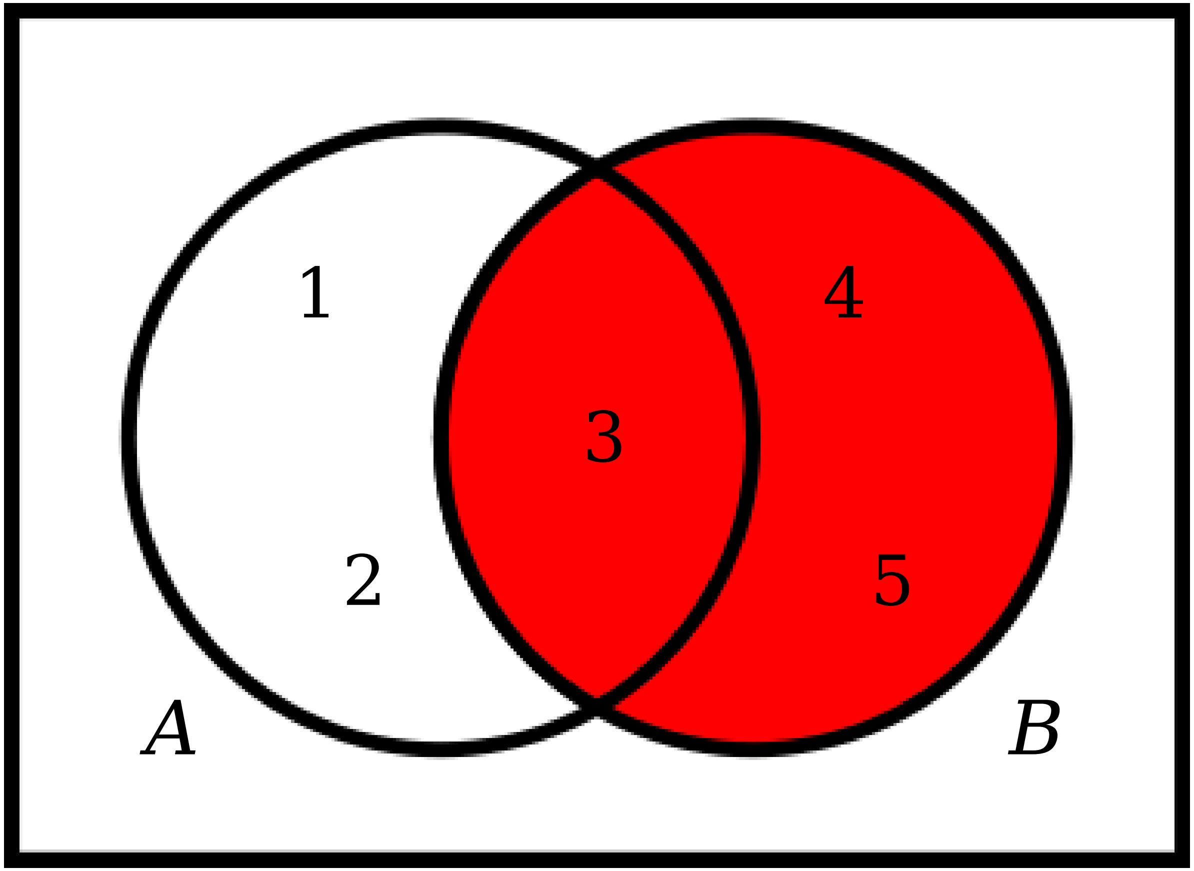
Venn Diagrams



A

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

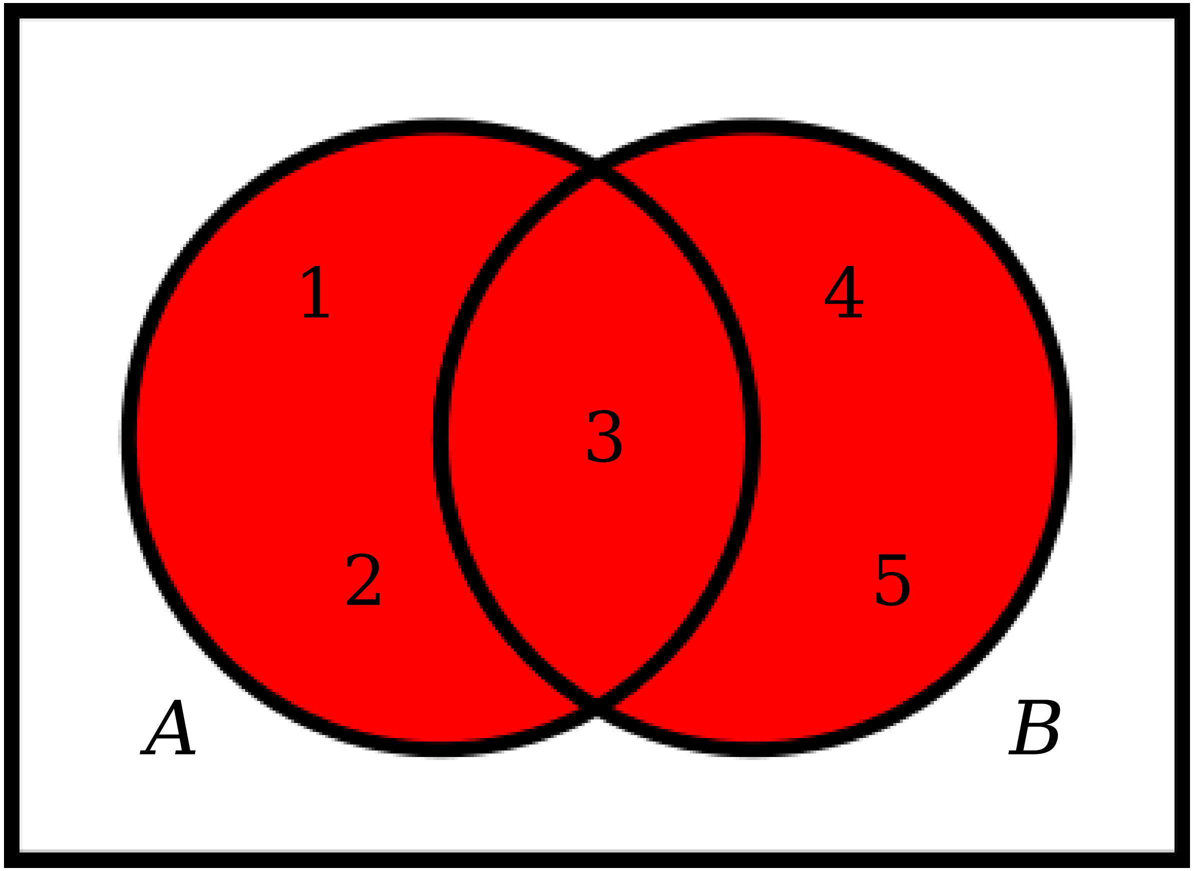
Venn Diagrams



B

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



Union

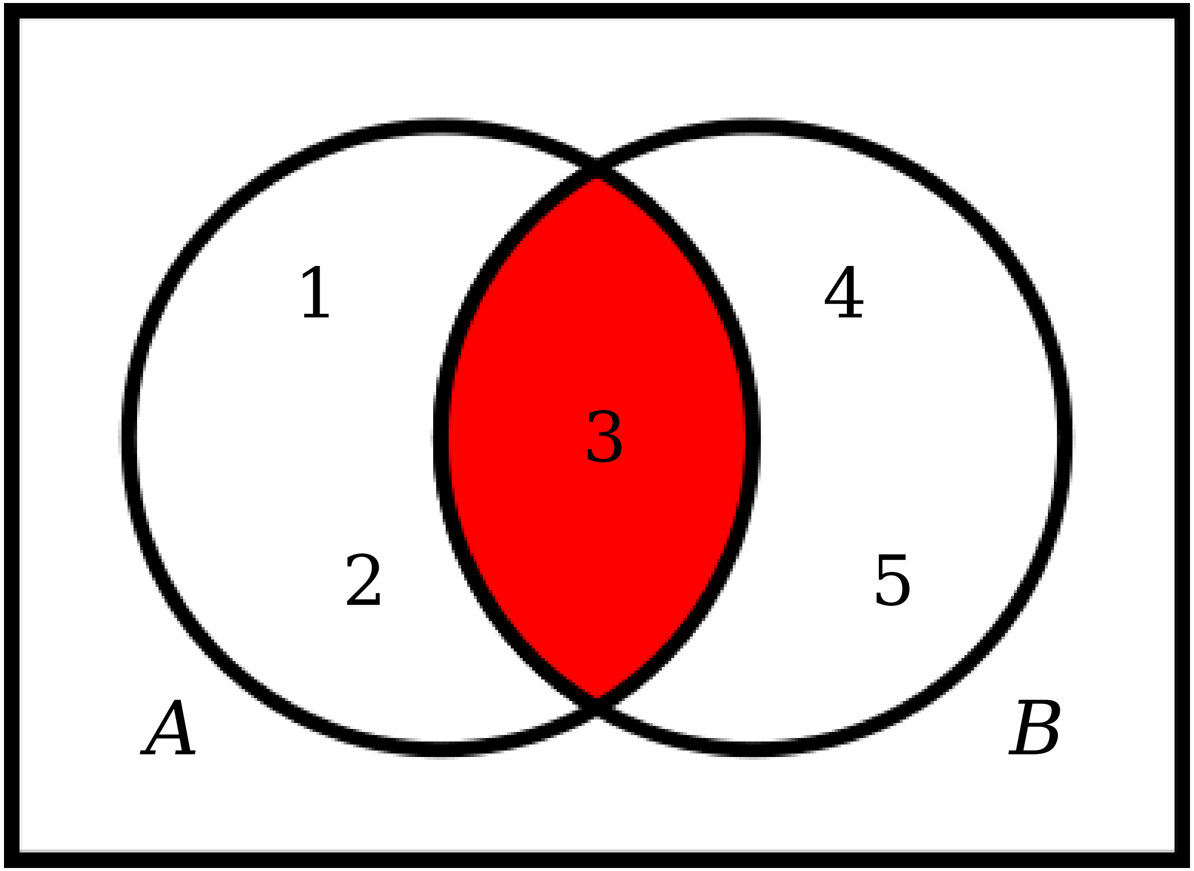
$A \cup B$

$\{ 1, 2, 3, 4, 5 \}$

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



Intersection

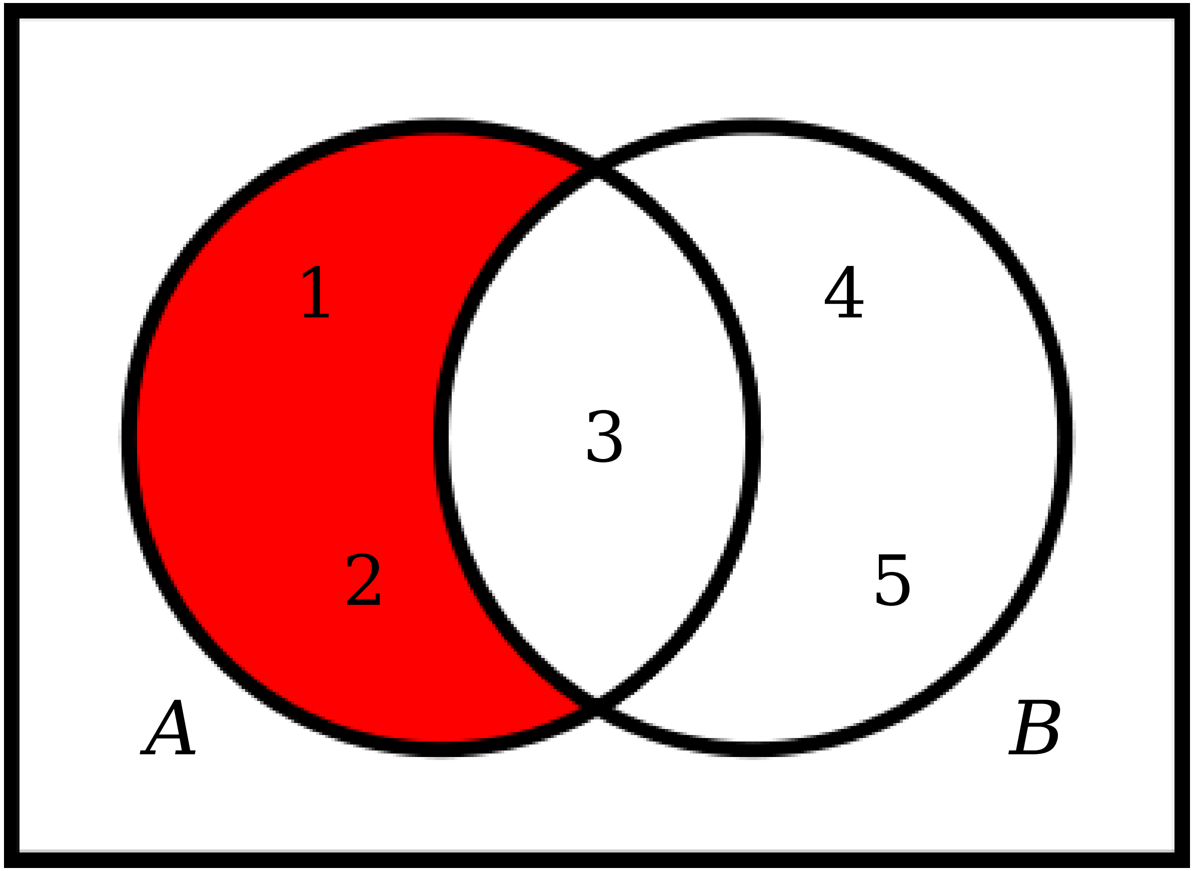
$A \cap B$

$\{ 3 \}$

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

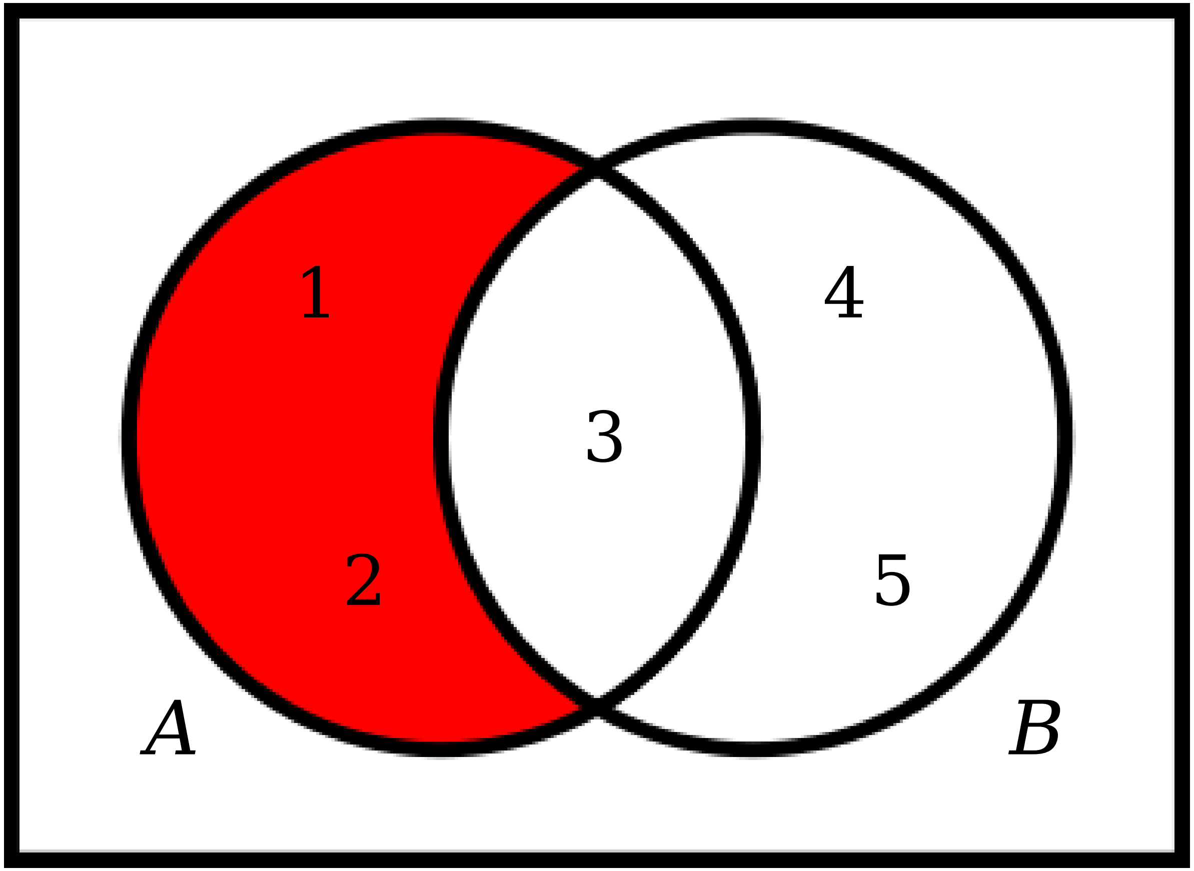
$A - B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

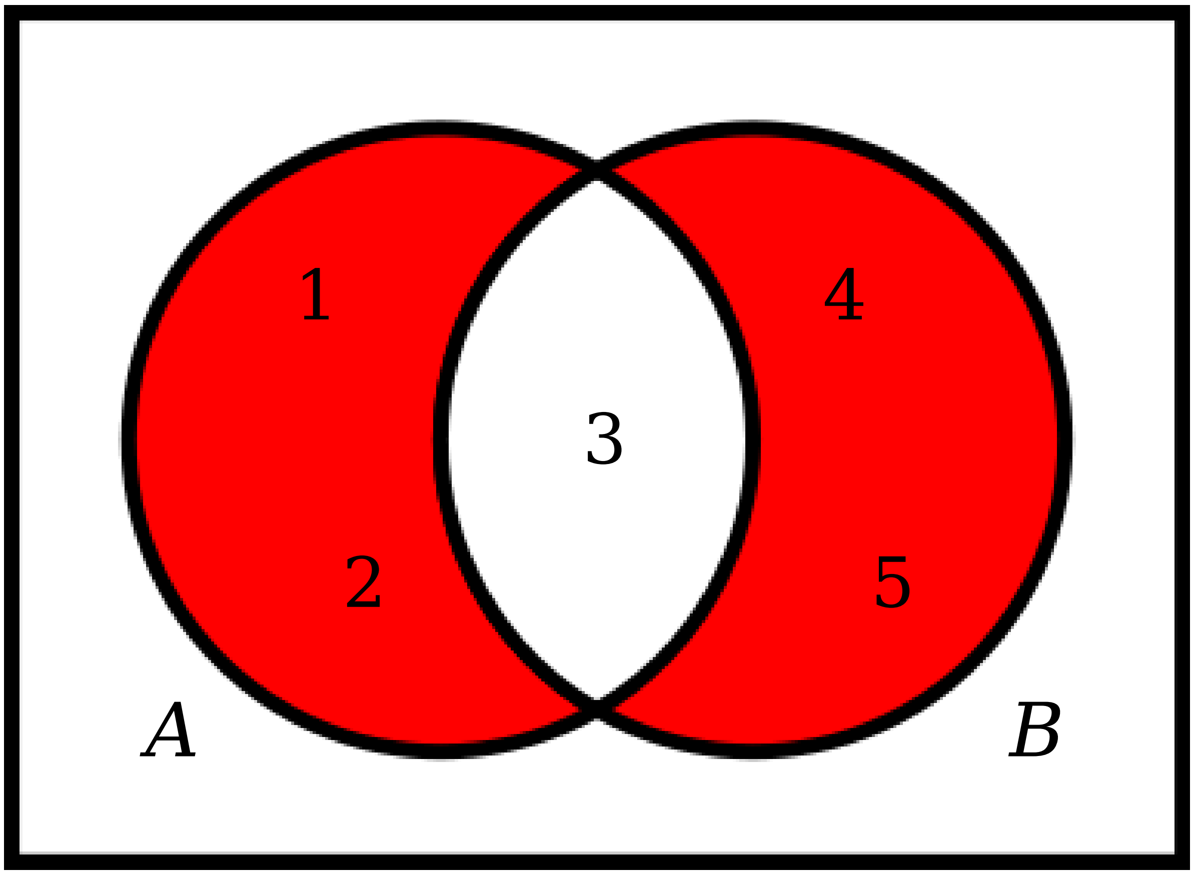
$A \setminus B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



Symmetric Difference

$A \, \Delta \, B$

$\{ 1, 2, 4, 5 \}$

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

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Power Set

$S = \{$ ,  $\}$

$\wp(S) = \{ \emptyset,$ $\{$  $\}$, $\{$  $\}$, $\{$ ,  $\}$ $\}$

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

The ***power set*** of S is the set of all subsets of S.

We write the power set of S as $\wp(S)$.

Power Set

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

The ***power set*** of S is the set of all subsets of S.

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Cardinality

- The ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality as $|S|$.
- Examples:
  - $|\{ whimsy, mirth \}| = 2$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 4 \}| = |\{0, 1, 2, 3\}| = 4$
  - $|\ \emptyset\ | = 0$
  - $|\ \{\emptyset\}\ | = 1$

The Cardinality of $\mathbb{N}$

- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define $\aleph_0 = |\mathbb{N}|$.
  - $\aleph_0$ is pronounced “aleph-zero,” “aleph-nought,” or “aleph-null.”
- ***Question:*** Why don’t we say $|\mathbb{N}| = \infty$?

Cardinality of Infinite Sets

Astonishing Fact:

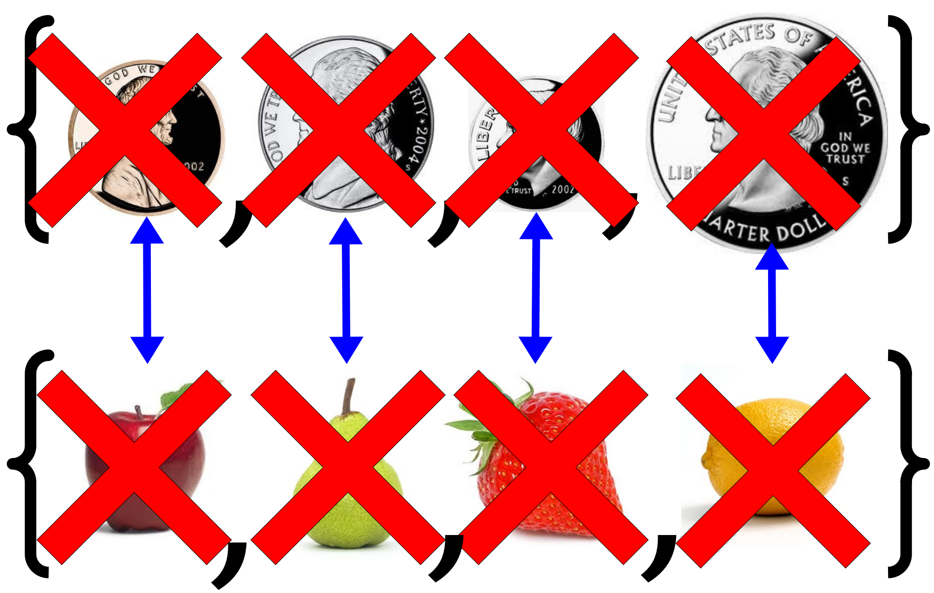
Not all infinite sets have the same cardinality.
Some infinite sets are bigger than others!

More Astonishing Fact:

This has practical consequences!

Comparing Cardinalities

- ***Question:*** What does it mean for one set to be "bigger" than another?
- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



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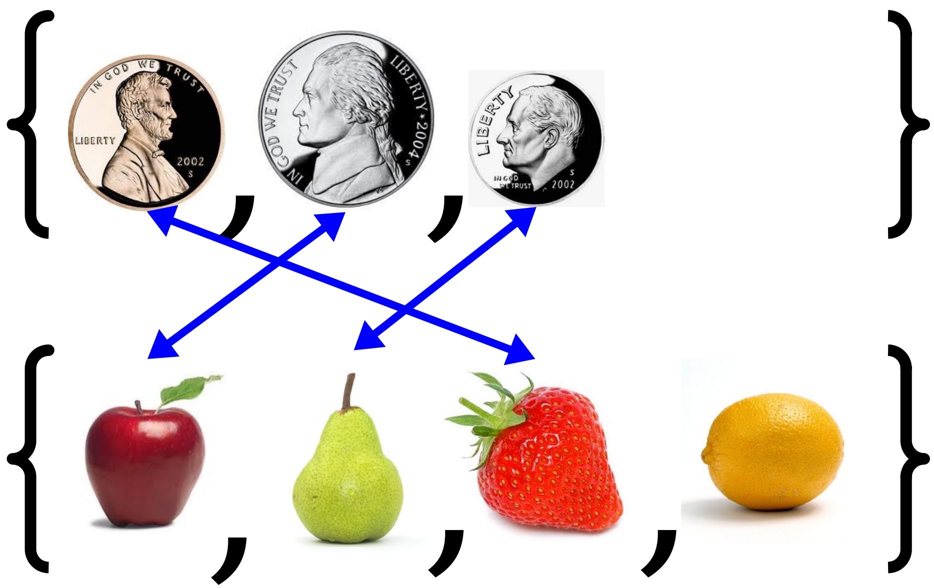
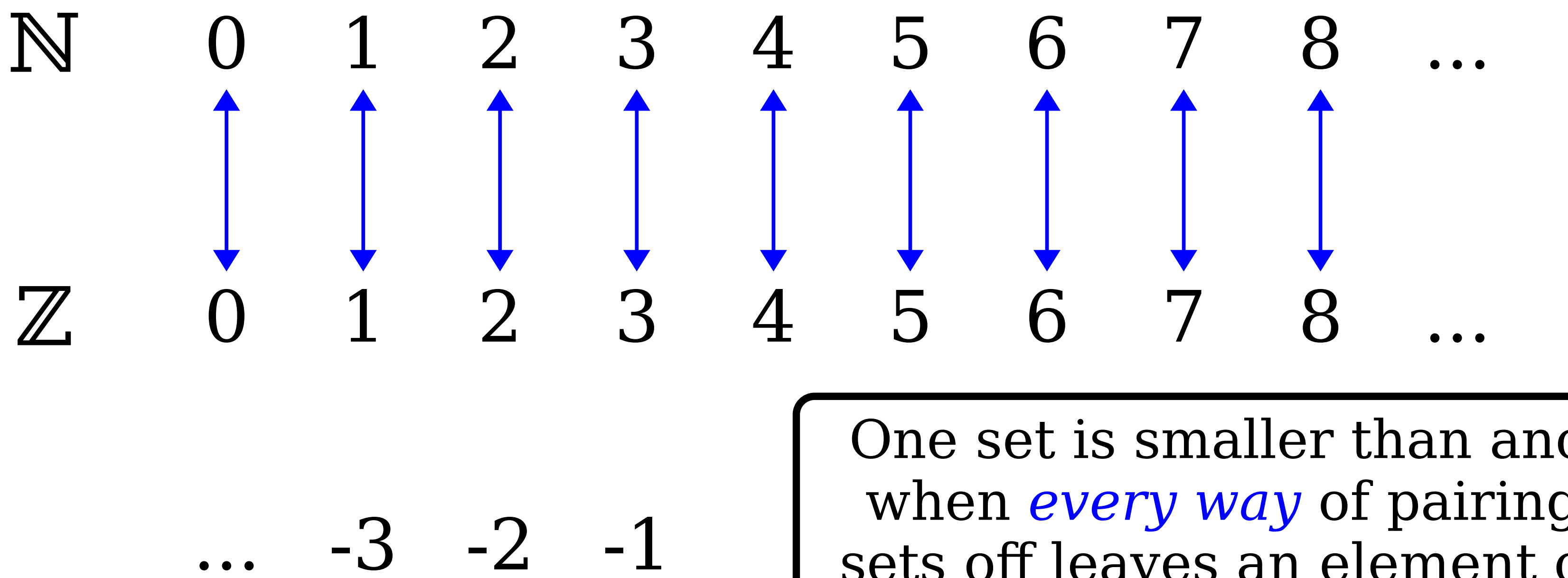


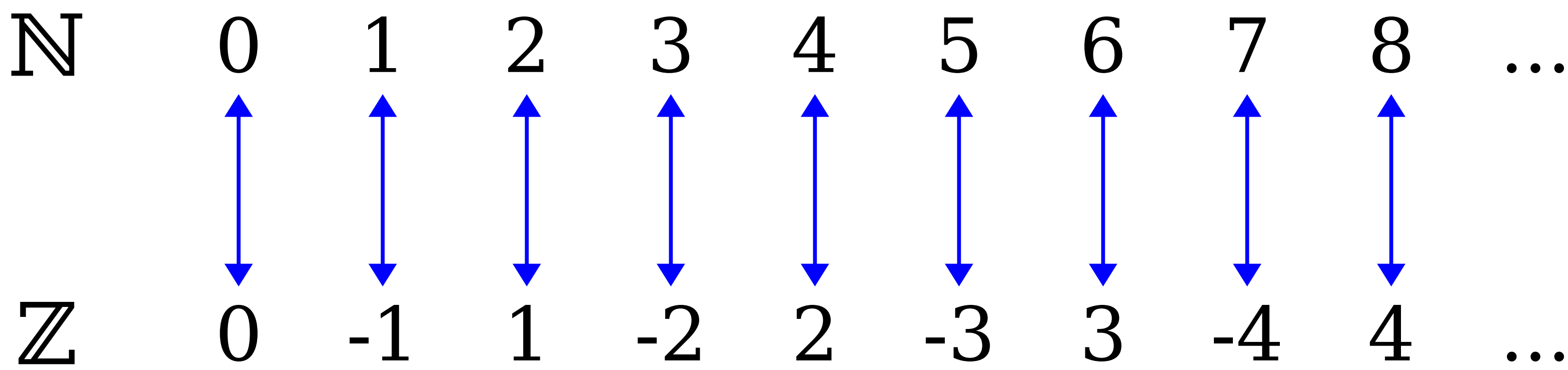
Infinite Cardinalities



One set is smaller than another when *every way* of pairing the sets off leaves an element of the second set uncovered.

Two sets have the same size when *there is a way* to pair their elements off without leaving any elements uncovered

Infinite Cardinalities



$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

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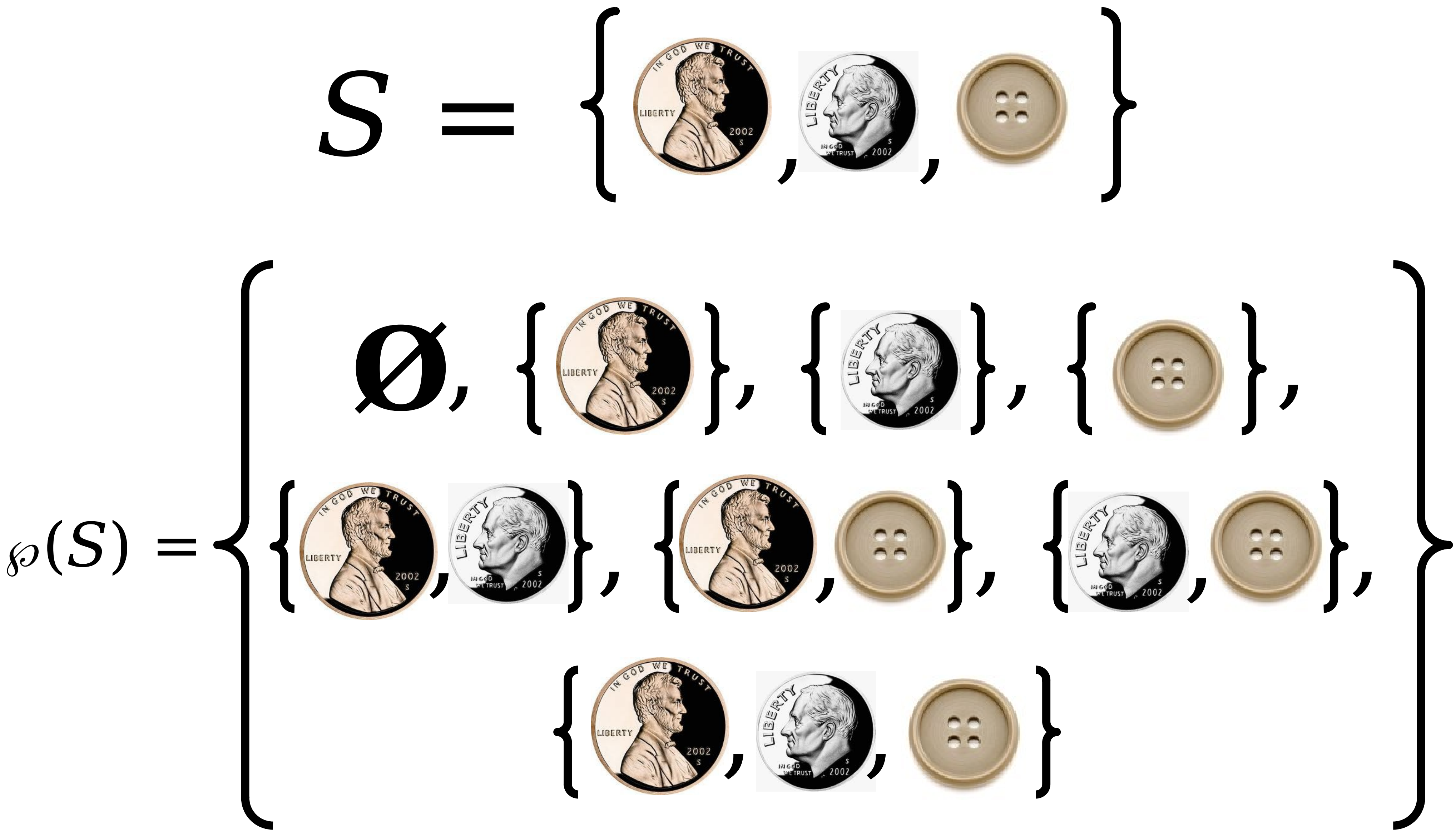
Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

$S = \{$ ,  $\}$

$\wp(S) = \{\ \varnothing,$ $\{$  $\}$, $\{$  $\}$, $\{$ ,  $\}\ \}$

$$|S| < |\wp(S)|$$



$$|S| < |\wp(S)|$$

$$S = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$$

$$\wp(S) = \Big\{ \varnothing,$$
$$\{\boldsymbol{a}\}, \{\boldsymbol{b}\}, \{\boldsymbol{c}\}, \{\boldsymbol{d}\},$$
$$\{\boldsymbol{a}, \boldsymbol{b}\}, \{\boldsymbol{a}, \boldsymbol{c}\}, \{\boldsymbol{a}, \boldsymbol{d}\}, \{\boldsymbol{b}, \boldsymbol{c}\}, \{\boldsymbol{b}, \boldsymbol{d}\}, \{\boldsymbol{c}, \boldsymbol{d}\}$$
$$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}, \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}, \{\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}\}, \{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\},$$
$$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}\Big\}$$

$$\boldsymbol{|S| < |\wp(S)|}$$

Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

Stated differently: no matter how you pair off the elements of a set S with the subsets of S, there is always some subset of S left uncovered.

Cantor’s Theorem

Consider the set

S = $\{x_0, x_1, x_2, x_3, x_4, x_5, \ldots\}$.

Let’s try pairing elements of S to elements of $\wp(S)$ and see what happens.

$$x_0 \leftrightarrow \{ x_0,\ x_2,\ x_4,\ \dots \}$$

$$x_1 \leftrightarrow \{ x_3,\ x_5,\ \dots \}$$

$$x_2 \leftrightarrow \{ x_0,\ x_1,\ x_2,\ x_5,\ \dots \}$$

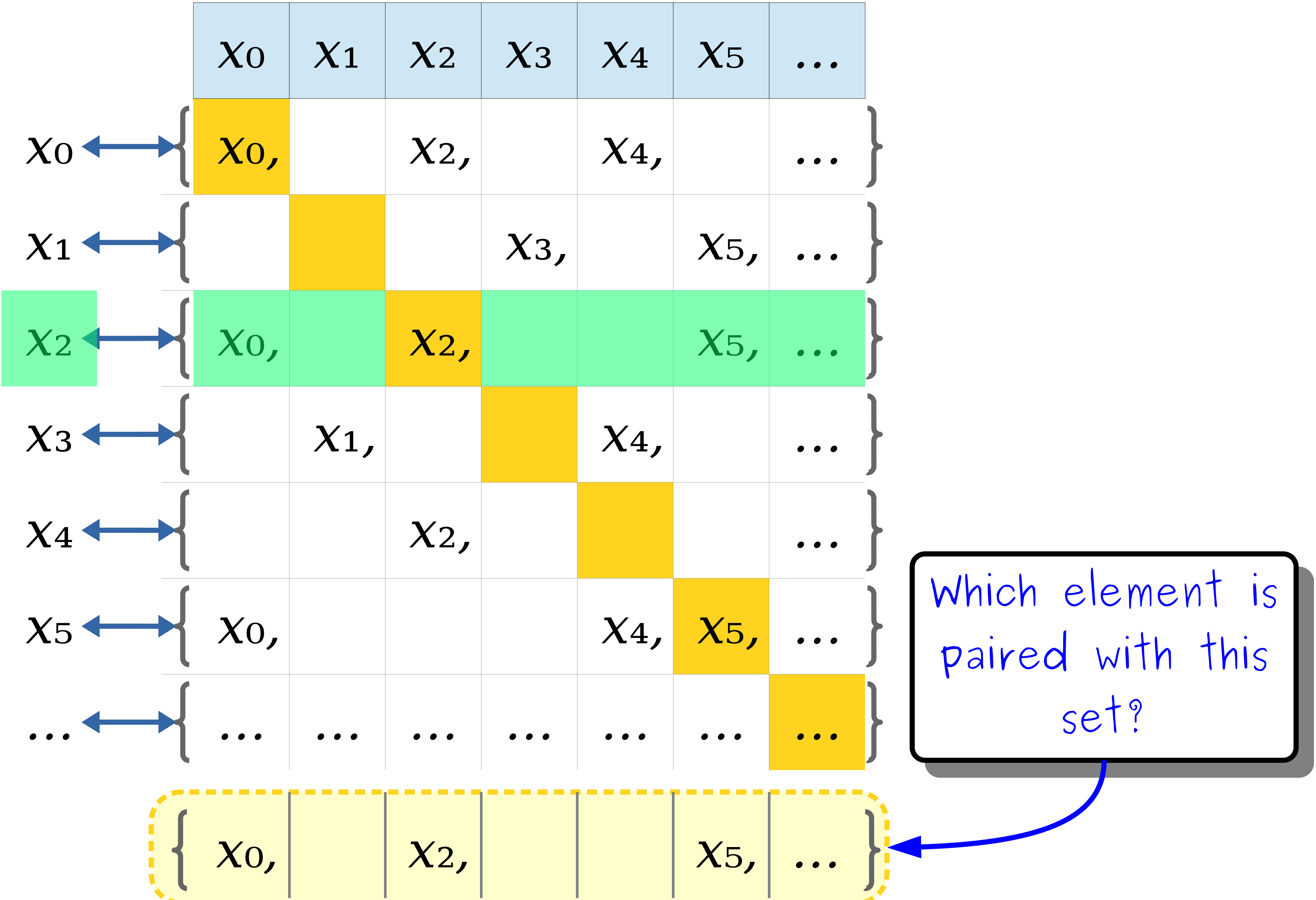
$$x_3 \leftrightarrow \{ x_1,\ x_4,\ \dots \}$$

$$x_4 \leftrightarrow \{ x_2,\ \dots \}$$

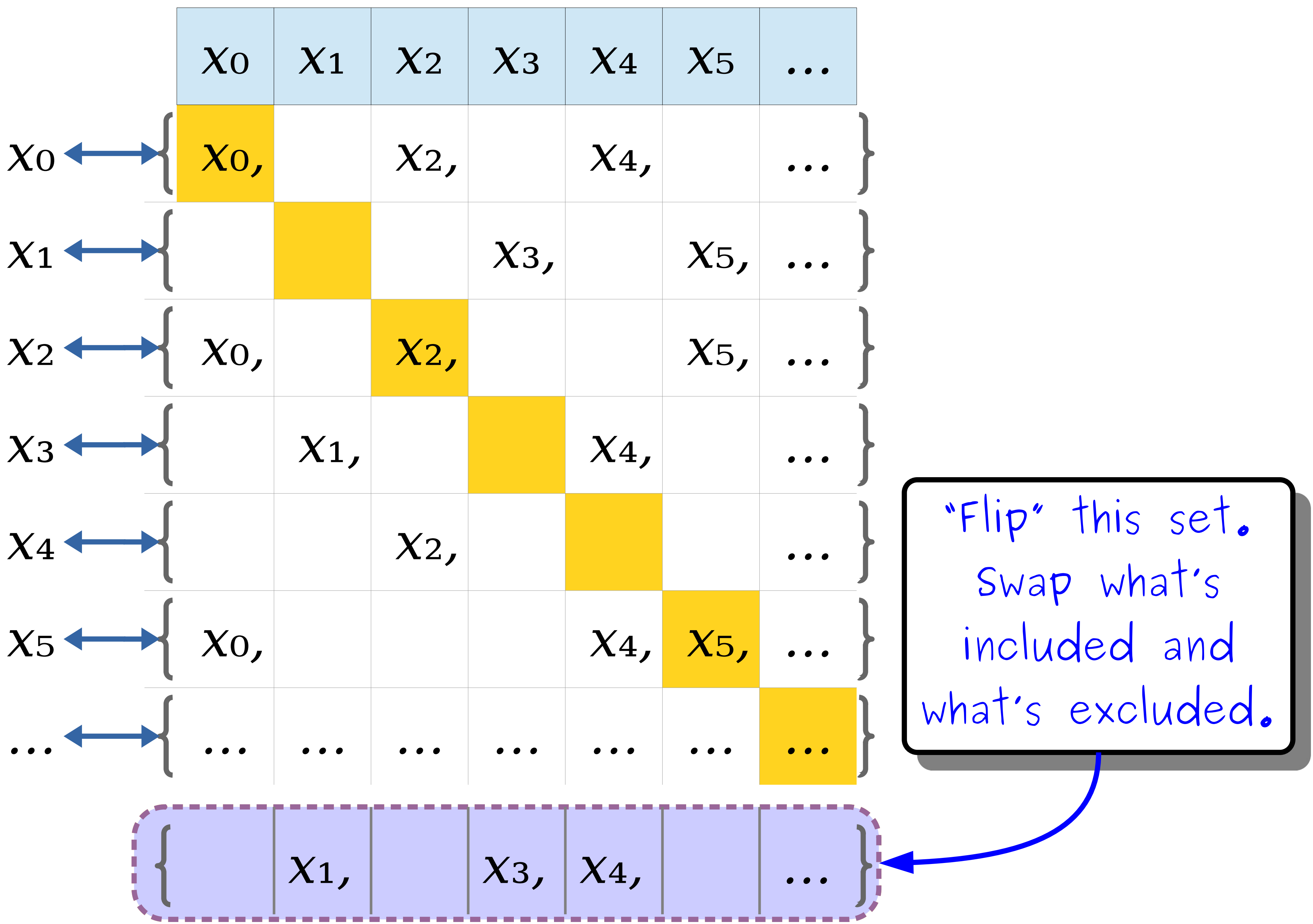
$$x_5 \leftrightarrow \{ x_0,\ x_4,\ x_5,\ \dots \}$$

$$\dots \leftrightarrow \{ \dots \}$$

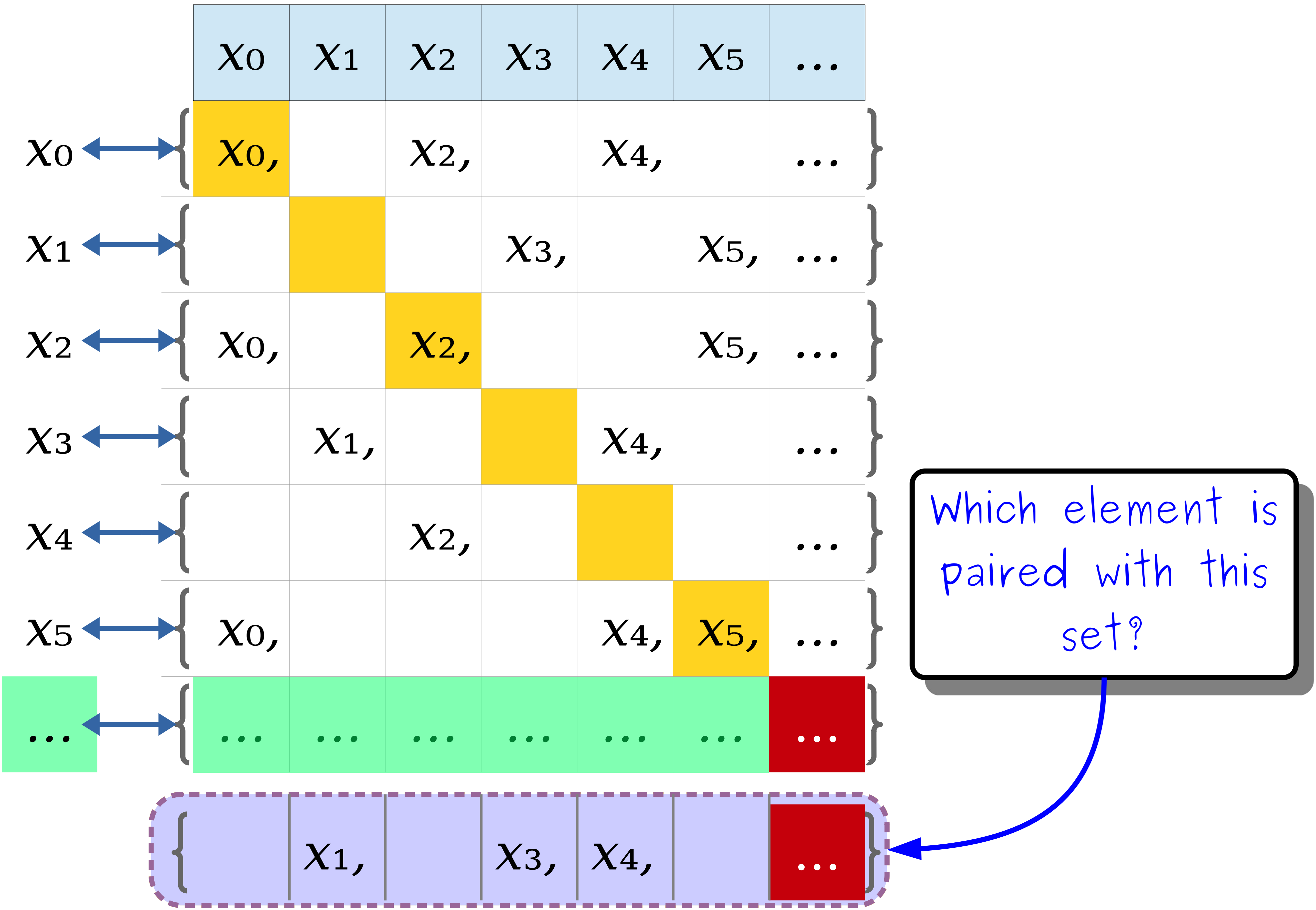
	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	{ x_0,		x_2,		x_4,		... }
x_1 ↔	{			x_3,		x_5,	... }
x_2 ↔	{ x_0,		x_2,			x_5,	... }
x_3 ↔	{	x_1,			x_4,		... }
x_4 ↔	{		x_2,				... }
x_5 ↔	{ x_0,				x_4,	x_5,	... }
... ↔	{ ...	...	...	...	...	...	... }



x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_0, x_2, x_5, ... }
x_3 { x_1, x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5, ... }
... { ... ... }
{ x_0, x_2, x_5, ... }
Which element is paired with this set?



x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x1, x3, x4, ... }
"Flip" this set.
Swap what's
included and
what's excluded.


x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2, x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x1, x3, x4, ... }
Which element is paired with this set?

… and Beyond!

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$

$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$

$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$

…

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor’s Theorem
9. **Shocking Implications**

Implications

How does this have any practical consequences?

What does this have to do with computation?

Let's examine two sets:

1. ***"The set of all computer programs"***

2. ***"The set of all problems to solve"***

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings *(see appendix!)*.

$$|\textbf{Programs}| \leq |\textbf{Strings}| < |\wp(\textbf{Strings})| \leq |\textbf{Problems}|$$

There are more problems to solve than there are programs to solve them!

$|\text{Programs}| < |\text{Problems}|$

It Gets Worse

- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
- In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.
- ***More troubling fact:*** We've just shown that *some* problems are impossible to solve with computers, but we don't know *which* problems those are!
- ***Key goal:*** We need to develop a more nuanced understanding of computation.

Where We're Going

- ***What makes a problem impossible to solve with computers?***
  - Is there a deep reason why certain problems can't be solved with computers, or is it completely arbitrary?
  - How do you know when you're looking at an impossible problem?
  - Are these real-world problems, or are they highly contrived?

Where We're Going

- ***How do we know that we're right?***
  - How can we back up our pictures with rigorous proofs?
  - How do we build a mathematical framework for studying computation?

Next Time

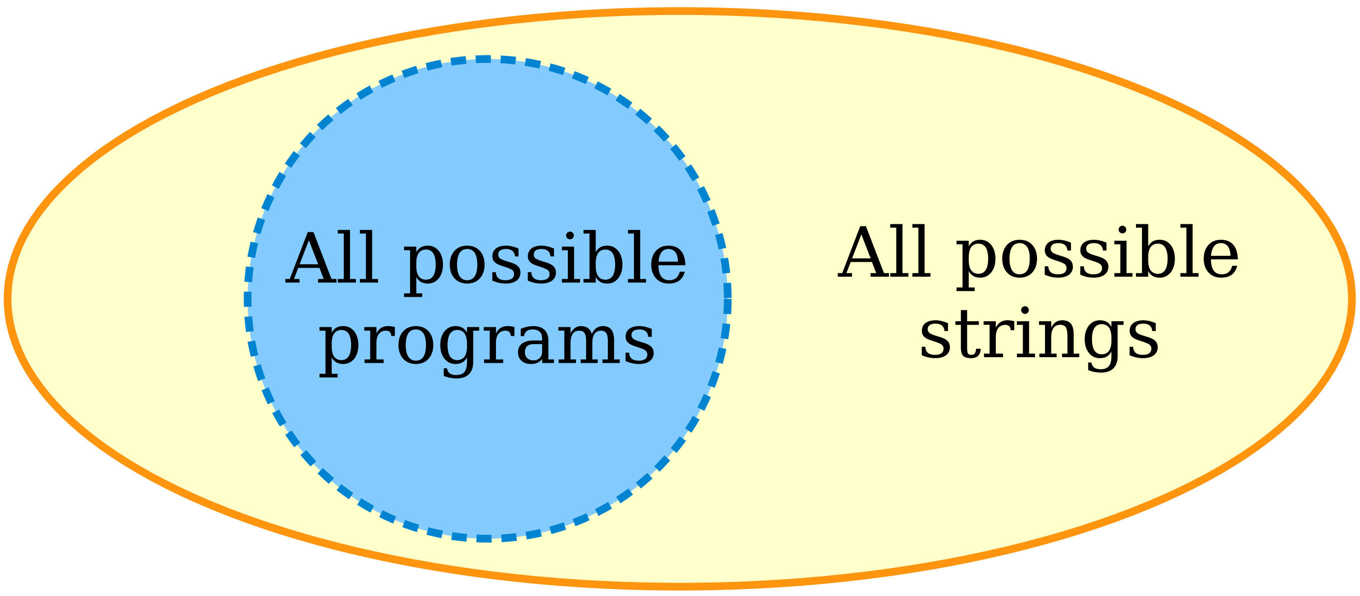
- ***Mathematical Proof***
  - What is a mathematical proof?
  - How can we prove things with certainty?

See also: Appendix: Strings and Problems

Appendix: Strings and Problems

Strings and Programs

- The source code of a computer program is just a (long, structured, well-commented) string of text.
- All programs are strings, but not all strings are necessarily programs.



|Programs| ≤ |Strings|

Strings and Problems

- There is a connection between the number of sets of strings and the number of problems to solve.
- Let S be any set of strings. This set S gives rise to a problem to solve:

Given a string w, determine whether $w \in S$.

Strings and Problems

Given a string *w*, determine whether $w \in S$.

- Suppose that S is the set

$$S = \{ \text{"}a\text{"}, \text{"}b\text{"}, \text{"}c\text{"}, \ldots, \text{"}z\text{"} \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether
***w* is a single lower-case English letter.**

Strings and Problems

Given a string *w*, determine whether *w* ∈ *S*.

- Suppose that *S* is the set
  - *S* = { "0", "1", "2", …, "9", "10", "11", … }
- From this set *S*, we get this problem:

Given a string w, determine whether w represents a natural number.

Strings and Problems

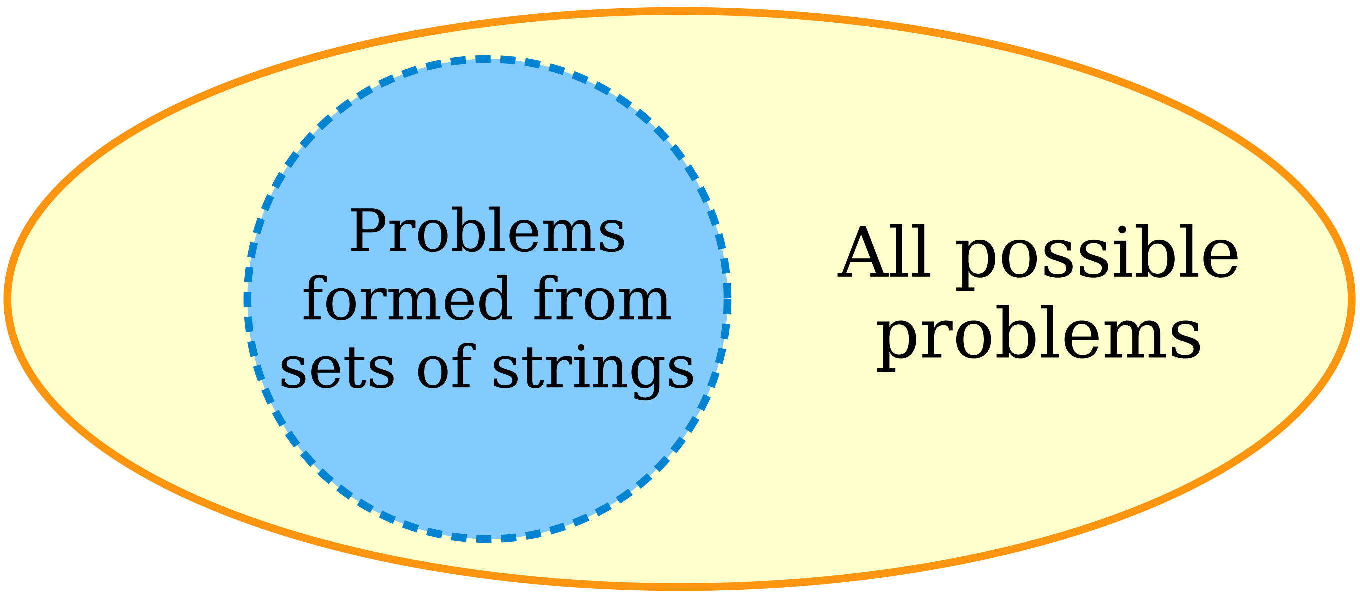
Given a string *w*, determine whether *w* ∈ *S*.

- Suppose that *S* is the set
  - *S* = { p | p is a legal C++ program }
- From this set *S*, we get this problem:

Given a string w, determine whether w is a legal C++ program.

Strings and Problems

- Every set of strings gives rise to a unique problem to solve.
- Other problems exist as well.



$|\wp(\text{Strings})| \leq |\text{Problems}|$